Question 1 [15 points]: Differentiate the following functions (you do not have to simplify your answers, however points will be deducted for improper use of notation):

(a) [3] $y = 3x^4 - \frac{\sqrt{x}}{2} - \frac{7}{x} = 3x^4 - \frac{1}{2}x^{\frac{1}{2}} - 7x^{-1}$

$y' = 12x^3 - \frac{1}{4}x^{-\frac{1}{2}} + 7x^{-2}$

(b) [3] $f(x) = \tan x \ln x$

$f'(x) = \sec^2 x \cdot \ln x + \tan x \cdot \frac{1}{x}$

(c) [3] $g(x) = \frac{3^x}{x + e^x}$

$$g'(x) = \frac{(x+e^x)\, 3^x \ln 3 - 3^x (1+e^x)}{(x+e^x)^2}$$

(d) [3] $f(x) = e^{\csc(x)}$

$f'(x) = e^{\csc(x)} \cdot [-\csc(x)\cot(x)]$

(e) [3] $y = \cos(\sqrt{1-x^2}) = \cos\left[(1-x^2)^{\frac{1}{2}}\right]$

$y' = -\sin\left[(1-x^2)^{\frac{1}{2}}\right] \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$

Question 2 [12 points]:

(a) [4] If $s(t) = t^3 - 3t + 1$ represents the position of a particle in metres at time $t \geq 0$ seconds, determine the acceleration of the particle when the velocity is 9 m/s.

$s'(t) = 3t^2 - 3$

$\therefore\ s'(t) = 9 \Rightarrow 3t^2 - 3 = 9 \Rightarrow t^2 = \frac{9+3}{3} = 4$

$\Rightarrow t = 2$ s.

$s''(t) = 6t\ ;\ s''(2) = 6(2) = 12\ \frac{m}{s^2}$

(b) [4] Compute $g''(\pi/4)$ if $g(x) = \ln(\sin x)$.

$g'(x) = \frac{1}{\sin x} \cdot (\cos x) = \frac{\cos x}{\sin x}$

$g''(x) = \frac{\sin x \cdot (-\sin x) - \cos x \cdot \cos x}{\sin^2 x} = \frac{-1}{\sin^2 x}$

$g''(\frac{\pi}{4}) = \frac{-1}{[\sin(\frac{\pi}{4})]^2} = \frac{-1}{\left(\frac{1}{\sqrt{2}}\right)^2} = -2$

(c) [4] Determine the point (x, y) on the graph of $y = \frac{e^x}{x}$ at which the tangent line is horizontal.

$y' = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$

$y' = 0$ at $x = 1$.

$\therefore$ Point is $x=1,\ y = \frac{e^1}{1} = e$, i.e. $(1, e)$.

Question 3 [12 points]: Evaluate the following limits (it may be useful to recall that $\lim_{x\to 0} \frac{\sin x}{x} = 1$):

(a) [3] $\lim_{x\to -4} \frac{x^2+x-12}{x^2+2x-8}$

$$= \lim_{x\to -4} \frac{(x-3)\cancel{(x+4)}}{(x-2)\cancel{(x+4)}}$$

$$= \frac{-7}{-6}$$

$$= \frac{7}{6}.$$

(b) [3] $\lim_{x\to\infty} \frac{-5x^7+7x^5}{7x^5-x-1}$ $\begin{matrix} \div x^5 \\ \div x^5 \end{matrix}$

$$= \lim_{x\to\infty} \frac{-5x^2+7}{7-\frac{1}{x^4}-\frac{1}{x^5}}$$

$$= -\infty.$$

(c) [3] $\lim_{x\to 0^+} \ln(1+\sqrt{x})$

as $x \to 0^+$, $\sqrt{x} \to 0^+$, so $1+\sqrt{x} \to 1^+$

$\therefore \ln(1+\sqrt{x}) \to 0$

$\therefore \lim_{x\to 0^+} \ln(1+\sqrt{x}) = 0$

(d) [3] $\lim_{\theta\to 0} \frac{\sin(3\theta)}{\tan(5\theta)}$

$$= \lim_{\theta\to 0} \frac{\sin(3\theta)}{(3\theta)} \cdot \frac{1}{\frac{\sin(5\theta)}{(5\theta)}} \cdot \cos(5\theta) \cdot \frac{3\theta}{5\theta}$$

$$= \frac{3}{5}$$

Question 4 [11 points]:

(a) [3] Determine the general antiderivative of $f(x) = x^{1/2} - \sec^2 x + \pi$

$$F(x) = \frac{2}{3} x^{3/2} - \tan x + \pi x + C$$

(b) [3] If $f'(x) = 2x - \frac{e^x}{2} - 1$ and $f(0) = -1$, determine $f(x)$.

$$f(x) = x^2 - \frac{e^x}{2} - x + C$$

$$f(0) = -1 \Rightarrow -1 = 0^2 - \frac{e^0}{2} - 0 + C$$

$$\Rightarrow C = \frac{-1}{2}$$

$$\therefore f(x) = x^2 - \frac{e^x}{2} - x - \frac{1}{2}$$

(c) [5] A particle has acceleration $a(t) = \sin t + \cos t$ where t is time in seconds. If the initial velocity is $v(0) = -1$ and initial position is $s(0) = 1$, determine the position of the particle at time $t = \pi$ seconds.

$$a(t) = \sin t + \cos t$$

$$\therefore v(t) = -\cos t + \sin t + C_1$$

$$v(0) = -1 \Rightarrow \cancel{-\cos(0)}^{-1} + \cancel{\sin(0)}^{0} + C_1 = -1$$

$$\Rightarrow C_1 = 0$$

$$\therefore v(t) = -\cos t + \sin t$$

$$\therefore s(t) = -\sin t - \cos t + C_2$$

$$s(0) = 1 \Rightarrow \cancel{-\sin(0)}^{0} \cancel{-\cos(0)}^{-1} + C_2 = 1$$

$$\Rightarrow C_2 = 2$$

$$\therefore s(t) = -\sin(t) - \cos(t) + 2$$

$$s(\pi) = \cancel{-\sin(\pi)}^{0} - \cancel{\cos(\pi)}^{-1} + 2 = 3$$

Question 5 [12 points]:

(a) [4] Determine the equation of the tangent line to $y = \dfrac{x + \ln x}{x^3}$ at the point where $x = 1$.

At $x=1$, $y = \dfrac{1+\ln 1}{1^3} = 1$

$$y' = \frac{x^3\left(1+\frac{1}{x}\right) - (x+\ln x)3x^2}{x^6}$$

$$y'\Big|_{x=1} = \frac{1^3\left(1+\frac{1}{1}\right) - (1+\ln 1)\,3\cdot 1^2}{1^6} = -1$$

$\therefore \quad y-1 = -1\,(x-1)$

or $\quad y = -x+2$

(b) [4] At the point where $x = a$ the tangent line to $f(x) = x^3$ is parallel to the tangent line to $g(x) = x^2 + x + 5$. Determine all possible values of a.

$$f'(a) = g'(a)$$

$$\therefore \; 3a^2 = 2a+1$$

$$3a^2 - 2a - 1 = 0$$

$$3a^2 - 3a + a - 1 = 0$$

$$3a(a-1) + (a-1) = 0$$

$$(3a+1)(a-1) = 0$$

$$3a+1 = 0 \qquad a-1 = 0$$

$$a = \frac{-1}{3}, \quad a = 1.$$

(c) [4] Use a linear approximation to estimate $\dfrac{1}{\sqrt{101}}$.

$f(x) = \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$, $a = 100$

$f(a) = \dfrac{1}{\sqrt{100}} = \dfrac{1}{10}$

$f'(x) = -\dfrac{1}{2}x^{-\frac{3}{2}} = \dfrac{-1}{2(\sqrt{x})^3}$

$f'(a) = \dfrac{-1}{2(\sqrt{100})^3} = \dfrac{-1}{2000}$

$\therefore \; f(x) \approx f(a) + f'(a)(x-a) = \dfrac{1}{10} + \left(\dfrac{-1}{2000}\right)(x-100)$

$\therefore \; f(101) \approx \dfrac{1}{10} - \dfrac{1}{2000}(101-100) = \dfrac{1}{10} - \dfrac{1}{2000} = \dfrac{199}{2000}$

Question 6 [8 points]:

(a) [4] For the curve defined by $\sqrt{xy} = x^2y - 2$, determine the equation of the tangent line at the point $(1, 4)$.

$$(xy)^{\frac{1}{2}} = x^2y - 2$$

$$\frac{1}{2}(xy)^{-\frac{1}{2}}[y + xy'] = 2xy + x^2y'$$

at $(1,4)$: $\frac{1}{2}(4)^{-\frac{1}{2}}[4 + y'] = 8 + y'$

$$\frac{1}{4}[4 + y'] = 8 + y'$$

$$4 + y' = 32 + 4y'$$

$$3y' = -28$$

$$y' = \frac{-28}{3}$$

$$\therefore \; y - 4 = \frac{-28}{3}(x - 1).$$

(b) [4] Use logarithmic differentiation to find y' if $y = (\sin x)^{\cos x}$.

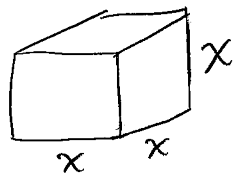
$$y = (\sin x)^{\cos x}$$

$$\ln y = \cos x \cdot \ln(\sin x)$$

$$\frac{1}{y}y' = -\sin x \cdot \ln(\sin x) + \cos x \cdot \frac{1}{\sin x} \cdot \cos x$$

$$\therefore y' = (\sin x)^{\cos x}\left[-\sin x \cdot \ln(\sin x) + \frac{\cos^2 x}{\sin x}\right].$$

Question 7 [8 points]: The volume of a melting cube of ice is decreasing by 2 cm^3/min. At what rate is the surface area of the cube decreasing when the side length of the cube is 4 cm? State units with your answer.



$S = 6x^2$

$\frac{dV}{dt} = -2 \ \frac{cm^3}{min}$

Find $\frac{dS}{dt}$ when $x = 4$.

$V = x^3$

$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{1}{3x^2} \frac{dV}{dt}$

$$\frac{dS}{dt} = 12x \frac{dx}{dt}$$

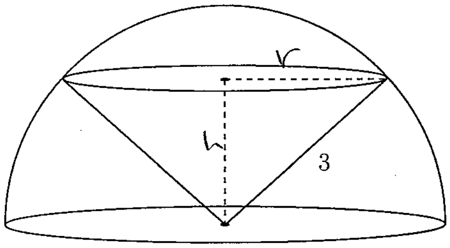
$$= 12x \cdot \frac{1}{3x^2} \frac{dV}{dt}$$

$$= \frac{4}{x} \frac{dV}{dt}$$

$\therefore$ When $x = 4$, $\frac{dS}{dt} = \frac{4}{4} \cdot (-2) = -2 \ \frac{cm^2}{min}$.

$\therefore$ The surface area is decreasing at $2 \ \frac{cm^2}{min}$.

Question 8 [10 points]: A right circular cone is inscribed in the upper half of a sphere of radius 3 m as shown. Find the largest possible volume of such a cone. Clearly justify all conclusions and state units with your answer. (Recall that the volume of a cone is $V = \pi r^2 h/3$.)



$r^2 + h^2 = 9$

$$V = \frac{\pi}{3} r^2 h$$

$$= \frac{\pi}{3}(9-h^2)h$$

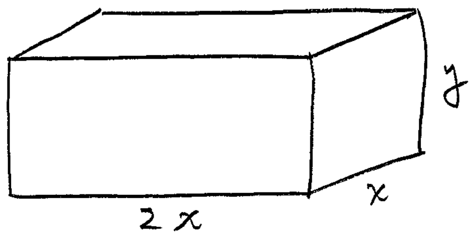
$$= \frac{\pi}{3}(9h - h^3), \qquad 0 \le h \le 3.$$

$$V' = \frac{\pi}{3}[9 - 3h^2]$$

$$= \pi[3-h^2] = 0 \quad \text{at} \quad h = \sqrt{3}$$

h	$V = \frac{\pi}{3}(9h-h^3)$
0	0
$\sqrt{3}$	$\frac{\pi}{3}(9\sqrt{3} - 3\sqrt{3}) = 2\pi\sqrt{3}$.
3	0

Question 9 [10 points]: A box has length equal to twice the width. The cost to ship the box is equal to the sum of the length, width and height. If the box must have a volume of 12 m^3, determine the dimensions (length, width and height) which minimize the shipping cost.



$$V = 2x^2y = 12 \text{ m}^3$$

$$C = 2x + x + y = 3x + y$$

minimize $C = 3x+y$ subject to $2x^2y = 12$.

$$2x^2y = 12 \Rightarrow y = \frac{12}{2x^2} = \frac{6}{x^2}$$

$\therefore$ minimize $C(x) = 3x + \frac{6}{x^2}$ on $(0,\infty)$.

$$C'(x) = 3 - \frac{12}{x^3} = \frac{3x^3-12}{x^3}.$$

$$C'(x) = 0 \Rightarrow 3x^3 = 12 \Rightarrow x^3 = 4 \Rightarrow x = 4^{1/3}$$

	1	$4^{1/3}$	4
$C'(x) = \frac{3x^3-12}{x^3}$	$-$	0	$+$
$C(x) = 3x + \frac{6}{x^2}$	$\searrow$		$\nearrow$

$\therefore$ Shipping cost is minimized if

width $x = 4^{1/3}$ m

length $2x = 2\cdot 4^{1/3} = 2^{5/3}$ m

height $y = \frac{6}{x^2} = \frac{6}{4^{2/3}} = \frac{3}{2^{1/3}}$ m

Question 10 [12 points]: For this question consider the function $f(x) = 20x^3 - 3x^5$.

(a) [4] Determine the intervals of increase and decrease of $f(x)$.

$$f'(x) = 60x^2 - 15x^4 = 15x^2(4-x^2) = 15x^2(2-x)(2+x)$$

$\therefore\ f'(x) = 0$ at $x = 0, 2, -2.$

	-3	-2	-1	0	1	2	3
$f'(x)$:	$-$	0	$+$	0	$+$	0	$-$
$f(x)$:	$\searrow$		$\nearrow$		$\nearrow$		$\searrow$

$\therefore$ f is increasing on $(-2, 0) \cup (0, 2)$,
decreasing on $(-\infty, -2) \cup (2, \infty)$.

(b) [2] State the x-coordinates of the relative extrema of $f(x)$.

f has a rel. min. at $x = -2$,
f has a rel. max. at $x = 2$.

(c) [4] Determine the intervals of concavity of $f(x)$.

$$f''(x) = 120x - 60x^3 = 60x(2-x^2)$$

$\therefore\ f''(x) = 0$ at $x = 0$, $x = \sqrt{2}$, $-\sqrt{2}$

	-2	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$	2
$f''(x)$:	$+$	0	$-$	0	$+$	0	$-$
$f(x)$:	$\cup$		$\cap$		$\cup$		$\cap$

$\therefore$ f is C.U. on $(-\infty, -\sqrt{2}) \cup (0, \sqrt{2})$;
f is C.D. on $(-\sqrt{2}, 0) \cup (\sqrt{2}, \infty)$.

(d) [2] State the x-coordinates of the inflection points of $f(x)$.

$x = -\sqrt{2}, 0, \sqrt{2}$.