Question 1:

(a)[3 points] Determine an equation of the line through the points $(-1, 4)$ and $(2, -5)$. (State your answer using any of the standard forms.)

$$m = \frac{-5-4}{2-(-1)} = \frac{-9}{3} = -3.$$

$$\therefore \quad y - 4 = -3\,(x - (-1))$$

$$\boxed{y - 4 = -3\,(x+1)}$$

or $\boxed{y = -3x + 1}$

(b)[4 points] Determine the slope and both the x and y intercepts of the line $7x - 3y = 11$.

$$7x - 3y = 11$$

$$-3y = -7x + 11$$

$$y = \frac{7}{3}x - \frac{11}{3}$$

at $y = 0$: $7x = 11$

$x = \frac{11}{7}$

$$\therefore \quad \boxed{m = \frac{7}{3}\,;\ x\text{-int}: \left(\frac{11}{7}, 0\right);\ y\text{-int}: \left(0, -\frac{11}{3}\right)}$$

(c)[3 points] Find an equation of the line through the point $(-3, -2)$ which is parallel to the line $5x + 10y = 1$. (State your answer using any of the standard forms.)

$$5x + 10y = 1$$

$$10y = -5x + 1$$

$$y = -\frac{1}{2}x + \frac{1}{10}$$

$$\therefore \quad m = \frac{-1}{2}$$

$$\therefore \quad \boxed{y + 2 = \frac{-1}{2}(x+3)} \quad \text{or} \quad \boxed{y = \frac{-1}{2}x - \frac{7}{2}}$$

Question 2:

(a) [5 points] The demand equation for a particular product is $D = 800 - 4p$, while the supply equation is $S = 20p - 1000$. Determine the quantity demanded at the market (or equilibrium) price.

$$800 - 4p = 20p - 1000$$

$$1800 = 24p$$

$$\therefore p = \frac{1800}{24} = 75$$

$$\therefore D = 800 - 4(75) = \boxed{500 \text{ units}}$$

(b) [5 points] An investor has \$20,000 to invest and two investments are available. The first pays 4% per year while a second riskier investment pays 6% per year. The investor's goal is to earn and withdraw \$1075 from the investment fund each year. How much interest is earned from the 4% investment during the first year? Round your answer to 2 decimals.

Let x = amount invested at 4%.
y = amount invested at 6%

$$\therefore x + y = 20{,}000 \Rightarrow y = 20{,}000 - x.$$

$$(0.04)x + (0.06)y = 1075$$

$$\therefore (0.04)x + (0.06)(20{,}000 - x) = 1075$$

$$(0.04)x - (0.06)x + (0.06)(20{,}000) = 1075$$

$$(-0.02)x = -125$$

$$x = \frac{125}{0.02} = \$6250$$

$\therefore$ Interest earned by 4% investment is $(0.04)(6250) = \boxed{\$250}$.

Question 3:

(a)[3 points] A system of linear equations has augmented coefficient matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & 0 \end{array}\right]$$

How many solutions does the system have? (exactly one, zero, or infinitely many).

$R_3 = 2r_2 + r_3$:
$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 6 \end{array}\right] \leftarrow \boxed{\text{no solution}}$$

(b)[3 points] Let $\mathbf{A} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$. Determine $\mathbf{A}^{-1}$

$$\left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]$$

$r_1 \leftrightarrow r_2$:
$$\left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 3 & -4 & 1 & 0 \end{array}\right]$$

$R_2 = (-3)r_1 + r_2$:
$$\left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{array}\right]$$

$R_2 = (-1)r_2$:
$$\left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$R_1 = r_2 + r_1$:
$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 4 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$\therefore \boxed{A^{-1} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}}$

(c)[4 points] Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 4 & 5 \\ 1 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$. Compute $(\mathbf{AB} + 2\mathbf{I_3})\,\mathbf{C}$.

$$AB = \begin{bmatrix} 4 & -5 & -3 \\ -10 & 1 & 19 \\ -2 & 0 & 4 \end{bmatrix}$$

$$AB + 2I_3 = \begin{bmatrix} 6 & -5 & -3 \\ -10 & 3 & 19 \\ -2 & 0 & 6 \end{bmatrix}$$

$$\therefore (AB + 2I_3)C = \begin{bmatrix} 6 & -5 & -3 \\ -10 & 3 & 19 \\ -2 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 16 \\ -16 \\ -2 \end{bmatrix}}$$

Question 4 [10 points]: Solve the following system of equations using matrix reduction:

$$\begin{aligned} 5x - 10y + 5z &= -15 \\ -5x + 8y - 7z &= -5 \\ 10x - 18y + 13z &= -3 \end{aligned} \quad : \left[\begin{array}{ccc|c} 5 & -10 & 5 & -15 \\ -5 & 8 & -7 & -5 \\ 10 & -18 & 13 & -3 \end{array}\right]$$

$R_1 = (\frac{1}{5})r_1$: $\left[\begin{array}{ccc|c} 1 & -2 & 1 & -3 \\ -5 & 8 & -7 & -5 \\ 10 & -18 & 13 & -3 \end{array}\right]$

$R_2 = 5r_1 + r_2$:
$R_3 = (-10)r_1 + r_3$: $\left[\begin{array}{ccc|c} 1 & -2 & 1 & -3 \\ 0 & -2 & -2 & -20 \\ 0 & 2 & 3 & 27 \end{array}\right]$

$R_2 = (\frac{-1}{2})r_2$: $\left[\begin{array}{ccc|c} 1 & -2 & 1 & -3 \\ 0 & 1 & 1 & 10 \\ 0 & 2 & 3 & 27 \end{array}\right]$

$R_3 = (-2)r_2 + r_3$: $\left[\begin{array}{ccc|c} 1 & -2 & 1 & -3 \\ 0 & 1 & 1 & 10 \\ 0 & 0 & 1 & 7 \end{array}\right]$

$\therefore\ z = 7$

$y = 10 - z = 10 - 7 = 3$

$x = -3 + 2y - z = -3 + 2(3) - 7 = -4$

$\therefore\ \boxed{x = -4,\ y = 3,\ z = 7}$

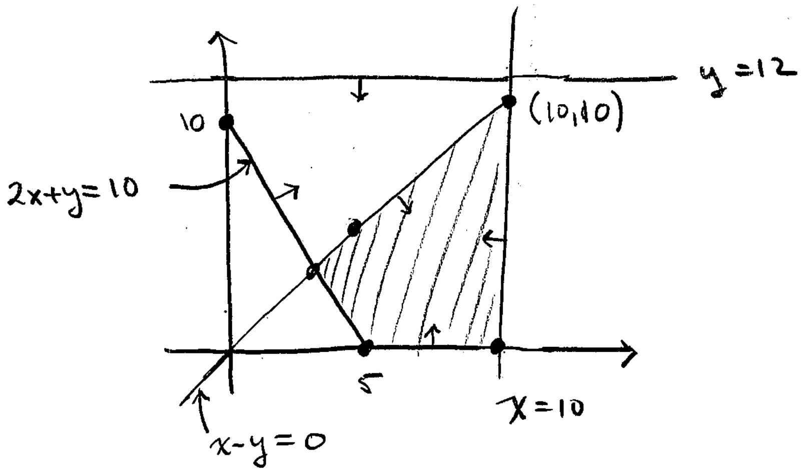
Question 5: [10 points] Maximize

$$z = 2x + 5y$$

subject to the constraints

$$\begin{aligned} x &\leq 10 \\ y &\leq 12 \\ 2x + y &\geq 10 \\ x - y &\geq 0 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Neatly draw any required graphs and show all work.



$$\left.\begin{aligned} 2x+y &= 10 \\ x-y &= 0 \end{aligned}\right\} \quad x=y, \quad 2x+x=10, \quad x=\frac{10}{3}, \; y=\frac{10}{3}.$$

C.P. (x,y)	$z=2x+5y$
(5,0)	$z=10$
(10,0)	$z=20$
(10,10)	$z=70$
$\left(\frac{10}{3},\frac{10}{3}\right)$	$z=2\left(\frac{10}{3}\right)+5\left(\frac{10}{3}\right)=\frac{70}{3}$

$\boxed{\therefore \text{ maximum value of } z \text{ is } 70.}$

Question 6: Round answers to 2 decimals.

(a)[2 points] \$400 is borrowed for 7 months at 8% simple interest. How much must be repaid at the end of the 7 months?

$$400\left(1+(0.08)\left(\tfrac{7}{12}\right)\right)=\boxed{\$418.67}$$

(b)[3 points] \$650 invested for 1.5 years at interest rate r compounded quarterly grows to \$710. What is the value of r?

$$710 = 650\left(1+\frac{r}{4}\right)^{(4)(1.5)}$$

$$\therefore\ r = 4\left[\left(\frac{710}{650}\right)^{\frac{1}{6}}-1\right] \approx 0.0593 = \boxed{5.93\%}$$

(c)[3 points] An investment earning interest compounded semiannually triples in 12 years. What is the interest rate?

$$P\left(1+\frac{r}{2}\right)^{(12)(2)} = 3P$$

$$r = 2\left(3^{\frac{1}{24}}-1\right) \approx \boxed{9.37\%}$$

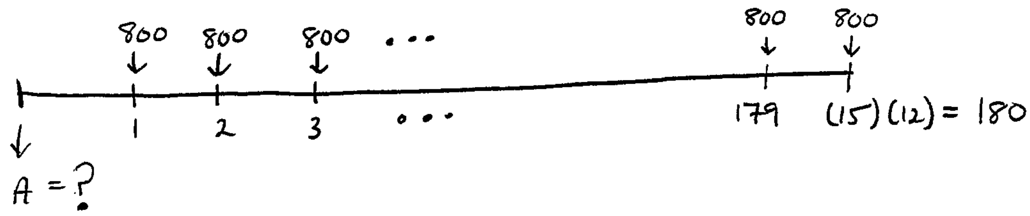
(d)[2 points] What is the effective rate of interest equivalent to 3.75% compounded monthly?

$$R = \left(1+\frac{0.0375}{12}\right)^{12}-1$$

$$\approx \boxed{3.82\%}$$

Question 7: Round answers to 2 decimals.

(a) [5 points] A loan will be repaid with payments of $800 made at the end of each month for 15 years. How much was the original loan amount if the rate of interest is 6% compounded monthly?



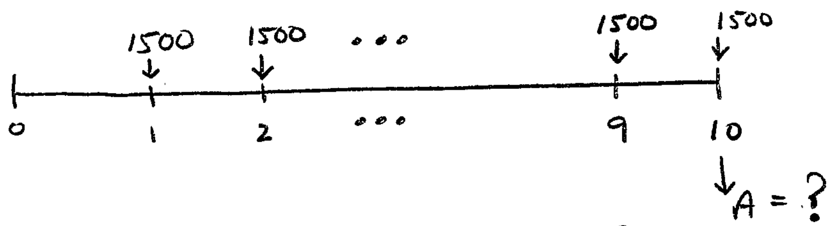
$$A = \frac{800}{\left(1+\frac{0.06}{12}\right)} + \frac{800}{\left(1+\frac{0.06}{12}\right)^2} + \cdots + \frac{800}{\left(1+\frac{0.06}{12}\right)^{180}}$$

$$= \frac{800}{1.005}\left[\frac{1-(1.005)^{-180}}{1-(1.005)^{-1}}\right]$$

$$\approx \boxed{\$94{,}802.81}$$

(b) [5 points] Suppose you are owed $10,000. The person who owes you the money offers to repay $1500 at the end of each year for 10 years, or you can elect to be paid a single payment of $20,000 at the end of the 10 years. If you plan to deposit the $1500 payments into an investment which pays 5% per year compounded annually, which of the two repayment options is better for you?

$r = 0.05$:



$$A = 1500 + 1500(1.05) + \cdots 1500(1.05)^9$$

$$= 1500\left[\frac{1-1.05^{10}}{1-1.05}\right]$$

$$= 18{,}866.84$$

$\boxed{\therefore \text{ The single } \$20{,}000 \text{ is better.}}$

Question 8:

(a)[4 points] Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and

$$A = \{0, 2, 4, 6, 8\}, \quad B = \{1, 3, 5, 7\}, \quad C = \{0\}$$

Determine

(i) $(B \cup C) \cap \bar{A}$

$B \cup C = \{0, 1, 3, 5, 7\}$

$(B \cup C) \cap \bar{A} = \boxed{\{1, 3, 5, 7\}}$.

(ii) $\overline{(\bar{A} \cup \bar{C})} = \overline{(\overline{A \cap C})} = A \cap C$

$= \boxed{\{0\}}$

(b)[4 points] Suppose $n(A) = 14$, $n(B) = 8$ and $n(A \cup B) = 14$.

(i) Determine $n(A \cap B)$.

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$14 = 14 + 8 - n(A \cap B)$

$\therefore n(A \cap B) = 14 + 8 - 14 = \boxed{8}$

(ii) Determine $n(A) - n(B)$.

Since $n(A \cup B) = n(A)$ then $B \subset A$.

$\therefore n(A) - n(B) = 14 - 8 = \boxed{6}$

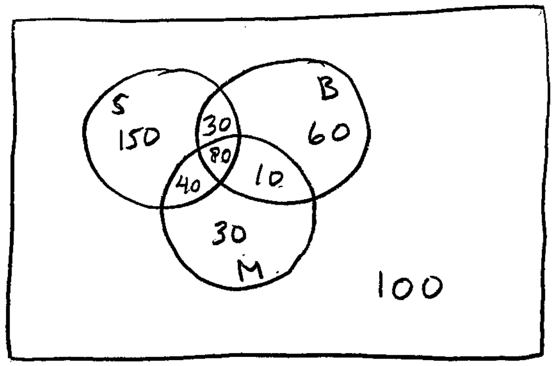
(c)[2 points] What is $\dfrac{C(1000, 998)}{P(1000, 998)}$? Give your answer in the form of a fraction.

$$\frac{\left(\dfrac{1000!}{(1000-998)!\, 998!}\right)}{\left(\dfrac{1000!}{(1000-998)!}\right)} = \frac{\cancel{1000!}}{\cancel{2!}\, 998!} \cdot \frac{\cancel{2!}}{\cancel{1000!}} = \boxed{\frac{1}{998!}}$$

Question 9:

(a) [4 points] From a group of 500 investors it was determined that

300 own stocks : S
180 own bonds : B
160 own mutual funds : M
110 own both stocks and bonds
120 own both stocks and mutual funds
90 own both bonds and mutual funds
80 own all three investment types



(i) How many own only bonds and mutual funds?

10

(ii) How many own none of these investments?

100

(b) [3 points] How many committees of 3 boys and 4 girls can be formed if 8 boys and 7 girls are available to serve?

$$C(8,3)\,C(7,4) = \frac{8!}{5!\,3!} \cdot \frac{7!}{4!\,3!} = \boxed{1960}$$

(c) [3 points] How many committees of 7 can be formed if 8 boys and 7 girls are available to serve and the committee must contain at least one member of each sex?

$$C(15,7) - C(8,7) - C(7,7)$$

$$= \frac{15!}{8!\,7!} - \frac{8!}{1!\,7!} - \frac{7!}{0!\,7!} = \boxed{6426}$$

Question 10:

(a)[3 points] How many distinct ordered arrangements of the letters KANAKANAK are possible? (KANAKANAK is a town in Alaska.)

$$\frac{9!}{3!\,4!\,2!} = \boxed{1260}$$

(b)[3 points] In how many ways can 5 girls and 3 boys be divided into two teams of four if each team must include at least one boy?

$$C(3,1)\,C(5,3)$$

$$= \frac{3!}{2!\,1!} \cdot \frac{5!}{2!\,3!}$$

$$= (3)(10)$$

$$= \boxed{30}$$

(c)[4 points] Determine the coefficient of x^6 in the expansion of $(x+3)^8$.

$$\binom{8}{2} x^6 (3)^2$$

$$= \left(\frac{8!}{6!\,2!} \cdot 3^2\right) x^6 = 252\,x^6$$

$\therefore$ coefficient of x^6 is $\boxed{252}$.

Question 11:

(a)[3 points] A person has probability 0.7 of passing math, 0.6 of passing physics, and 0.9 of passing at least one of the courses. Determine the probability of passing both courses.

M: pass math ; $P(M) = 0.7$

H: pass physics, $P(H) = 0.6$

$P(M \cup H) = 0.9$

$P(M \cap H) = ?$

$$P(M \cup H) = P(M) + P(H) - P(M \cap H)$$

$$0.9 = 0.7 + 0.6 - P(M \cap H)$$

$$\therefore P(M \cap H) = 0.7 + 0.6 - 0.9 = \boxed{0.4}.$$

(b)[4 points] The odds for Bart winning a race are 3 to 4, while the odds for Millhouse winning are 1 to 5. What are the odds for Bart or Millhouse winning the race, assuming a tie is impossible?

B : Bart wins , $P(B) = \frac{3}{3+4} = \frac{3}{7}$

M : Millhouse wins , $P(M) = \frac{1}{1+5} = \frac{1}{6}$.

$$P(B \cup M) = P(B) + P(M) - \cancel{P(B \cap M)}^{0}$$

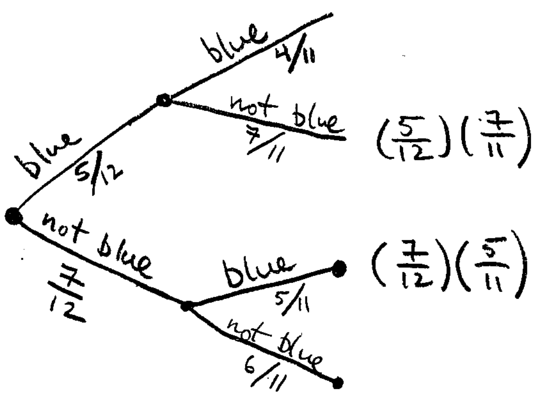
$$= \frac{3}{7} + \frac{1}{6}$$

$$= \frac{18+7}{42}$$

$$= \frac{25}{42} = \frac{25}{25+17}$$

$\boxed{\therefore \text{ odds are 25 to 17.}}$

(c)[3 points] A jar contains 4 white, 3 yellow and 5 blue marbles. Two marbles are chosen from the jar without replacement. What is the probability that exactly one marble is blue?



$$\therefore P = \left(\frac{5}{12}\right)\left(\frac{7}{11}\right) + \left(\frac{7}{12}\right)\left(\frac{5}{11}\right)$$

$$= 2\left(\frac{35}{132}\right)$$

$$= \boxed{\frac{35}{66}}$$

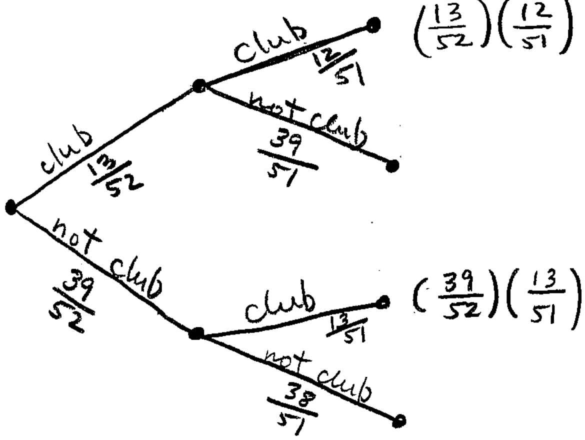
Question 12:

(a)[3 points] In a room of 5 people, what is the probability that at least two people share the same month of birth? (Here "month of birth" means January, February, March, ..., December.)

E : at least two people share same month of birth
$\bar{E}$: all 5 people have different months of birth.

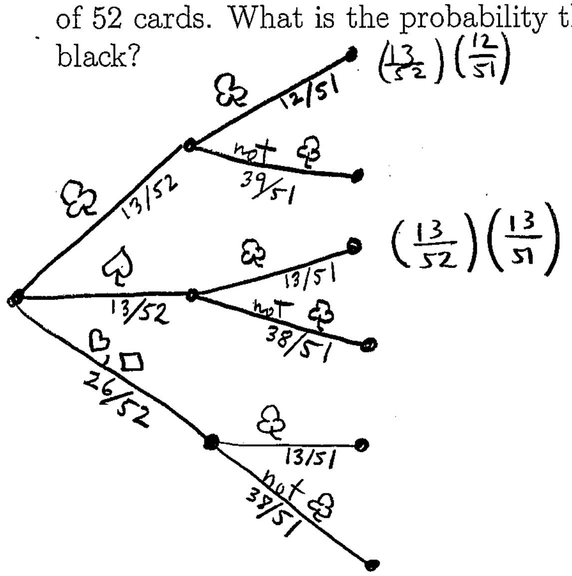
$$P(E) = 1 - P(\bar{E})$$
$$= 1 - \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{12^5}$$
$$\approx \boxed{0.618}$$

(b)[3 points] Two cards are drawn, one after the other, without replacement from a deck of 52 cards. What is the probability that the second card is a ♣ (club)?



$$\therefore P = \left(\frac{13}{52}\right)\left(\frac{12}{51}\right) + \left(\frac{39}{52}\right)\left(\frac{13}{51}\right)$$
$$= \left(\frac{13}{52}\right)\left[\frac{12+39}{51}\right]$$
$$= \left(\frac{1}{4}\right)(1)$$
$$= \boxed{0.25}$$

(c)[4 points] Again two cards are drawn, one after the other, without replacement from a deck of 52 cards. What is the probability that the second card is a ♣ (club) given that the first is black?



E : 2^{nd} card is ♣
F : 1^{st} card is ♣ or ♠

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$
$$= \frac{\left(\cancel{\frac{13}{52}}\right)\left(\frac{12}{51}\right) + \left(\cancel{\frac{13}{52}}\right)\left(\frac{13}{51}\right)}{2\left(\cancel{\frac{26}{52}}\right)}$$
$$= \frac{25}{102} \approx \boxed{0.245}$$