Math 122 Sec F08N01 - Test 3
Nov 21 2008

Question 1:

(a)[5 points] Use T_6, the trapezoid rule on six subintervals, to approximate $\int_0^{3\pi} x^2 \cos x \, dx$.

$$\Delta x = \frac{3\pi}{6} = \frac{\pi}{2}$$

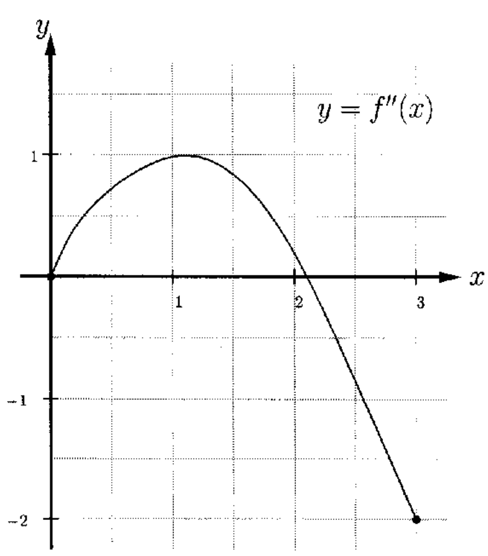
$$\therefore T_6 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_5) + f(x_6)\right]$$

$$= \frac{\pi}{4}\Big[0^2\cancelto{0}{\cos(0)} + 2\left(\tfrac{\pi}{2}\right)^2\cancelto{0}{\cos\left(\tfrac{\pi}{2}\right)} + 2(\pi)^2\cos(\pi) + \cancelto{0}{2\left(\tfrac{3\pi}{2}\right)^2\cos\left(\tfrac{3\pi}{2}\right)}$$
$$+ 2(2\pi)^2\cos(2\pi) + \cancelto{0}{2\left(\tfrac{5\pi}{2}\right)^2\cos\left(\tfrac{5\pi}{2}\right)} + (3\pi)^2\cos(3\pi)\Big]$$

$$= \frac{\pi}{4}\left[-2\pi^2 + 8\pi^2 - 9\pi^2\right]$$

$$= \boxed{\frac{-3\pi^3}{4}}$$

(b)[5 points] The graph of $y = f''(x)$ is shown below. If the trapezoid rule is used to approximate $\int_0^3 f(x)\,dx$, how many subintervals are required to ensure that the error in our approximation is less than 1/16. Recall, the error in using the trapezoid rule to approximate $\int_a^b f(x)\,dx$ is at most $\frac{K(b-a)^3}{12n^2}$, where $|f''(x)| < K$ on $[a, b]$.



$\therefore |f''(x)| \leq 2$ on $[0,3]$,

so use $K = 2$.

Need $\frac{2(3-0)^3}{12n^2} < \frac{1}{16}$

$$\frac{2\cdot\cancelto{9}{27}\cdot\cancelto{4}{16}}{\cancel{12}\,\cancel{3}} < n^2$$

$$72 < n^2$$

$$\therefore 6\sqrt{2} < n$$

$$\therefore \boxed{n > 6\sqrt{2}}$$

(or $n \geq 9$)

Question 2:

(a)[5 points] Evaluate the improper integral or show that is it divergent.

$$I = \int_3^5 \frac{x}{\sqrt{x^2-9}}\,dx = \lim_{t\to 3^+} \int_t^5 \frac{x}{\sqrt{x^2-9}}\,dx$$

For $\int \frac{x}{\sqrt{x^2-9}}\,dx$, let $u = x^2-9$, $du = 2x\,dx$,

$\therefore \int \frac{x}{\sqrt{x^2-9}}\,dx = \frac{1}{2}\int u^{-\frac{1}{2}}\,du = u^{\frac{1}{2}} + C = \sqrt{x^2-9} + C.$

$\therefore I = \lim_{t\to 3^+} \left[\sqrt{x^2-9}\right]_t^5 = \lim_{t\to 3^+} \left[4 - \sqrt{t^2-9}\right] = 4$

$\therefore$ integral converges to 4.

(b)[5 points] Evaluate the improper integral or show that is it divergent.

$$\int_{-\infty}^{\infty} \frac{x^2}{1+3x^3}\,dx$$

$$= \int_{\infty}^{0} \frac{x^2}{1+3x^3}\,dx + \int_0^{\infty} \frac{x^2}{1+3x^3}\,dx.$$

For $\int \frac{x^2}{1+3x^3}\,dx$, let $u = 1+3x^3$, $du = 9x^2$,

$\therefore \int \frac{x^2}{1+3x^3}\,dx = \frac{1}{9}\int \frac{1}{u}\,du = \frac{1}{9}\ln|u| + C = \frac{1}{9}\ln|1+3x^3| + C.$

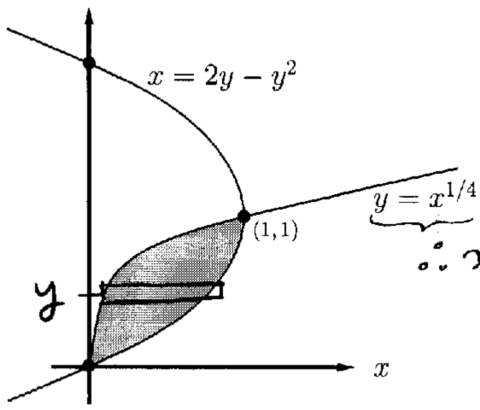
$\therefore \int_0^{\infty} \frac{x^2}{1+3x^3} = \lim_{t\to\infty} \int_0^t \frac{x^2}{1+3x^3}\,dx = \lim_{t\to\infty} \left[\frac{1}{9}\ln|1+3x^3|\right]_0^t$

$= \lim_{t\to\infty} \left[\frac{1}{9}\ln|1+3t^3| - 0\right] = \infty$

$\therefore$ Since $\int_0^{\infty} \frac{x^2}{1+3x^3}\,dx$ diverges, so does $\int_{-\infty}^{\infty} \frac{x^2}{1+3x^3}\,dx$.

Question 3:

(a)[5 points] Find the area of the shaded region below:



$\therefore x = y^4$

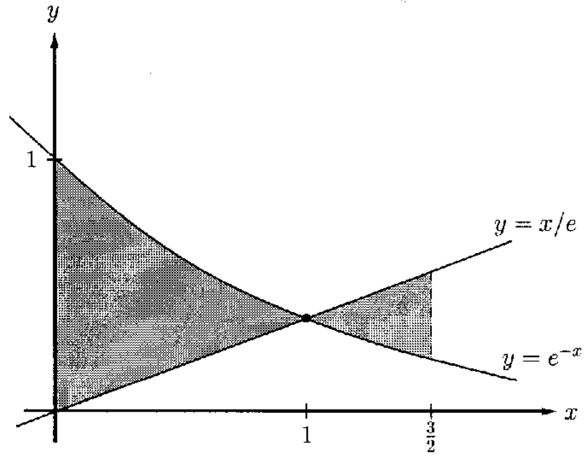
$$A = \int_0^1 2y - y^2 - y^4 \, dy$$

$$= \left[\frac{2y^2}{2} - \frac{y^3}{3} - \frac{y^5}{5} \right]_0^1$$

$$= \left(1 - \frac{1}{3} - \frac{1}{5}\right) - 0$$

$$= \frac{15-5-3}{15} = \boxed{\frac{7}{15}}$$

(b)[5 points] Find the area of the shaded region below:



$$A = \int_0^1 \left(e^{-x} - \frac{x}{e}\right) dx + \int_1^{3/2} \frac{x}{e} - e^{-x} \, dx$$

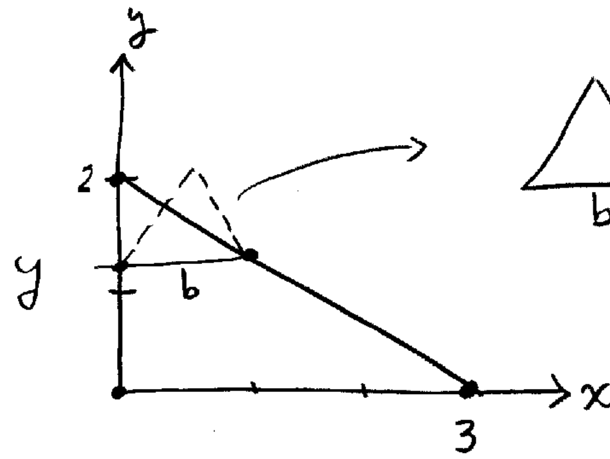
$$= \left[-e^{-x} - \frac{x^2}{2e} \right]_0^1 + \left[\frac{x^2}{2e} + e^{-x} \right]_1^{3/2}$$

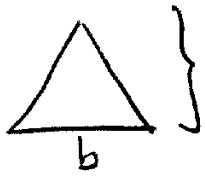
$$= -\frac{1}{e} - \frac{1}{2e} + 1 + 0 + \frac{9}{8e} + \frac{1}{e^{3/2}} - \frac{1}{2e} - \frac{1}{e}$$

$$= -\frac{8}{8e} - \frac{4}{8e} + 1 + \frac{9}{8e} + \frac{1}{e^{3/2}} - \frac{4}{8e} - \frac{8}{8e}$$

$$= \frac{-15}{8e} + \frac{1}{e^{3/2}} + 1$$

Question 4 [10 points]: The base (flat bottom surface) of the solid S is the triangular region with vertices at $(0, 0)$, $(3, 0)$ and $(0, 2)$. Cross-sections perpendicular to the y-axis are isosceles triangles of equal base and height. Find the volume of S.





$b \therefore A = \frac{1}{2} b^2$

Using similar triangles, $\frac{2-y}{2} = \frac{b}{3}$

$\therefore b = \frac{3}{2}(2-y).$

$$\therefore A(y) = \frac{1}{2}\left[\frac{3}{2}(2-y)\right]^2$$

$$= \frac{1}{2} \cdot \frac{9}{4}(2-y)^2$$

$$= \frac{9}{8}(2-y)^2$$

$$\therefore V = \int_0^2 \frac{9}{8}(2-y)^2\, dy$$

$$= -\frac{\cancel{9}^{3}}{8}\left[\frac{(2-y)^3}{\cancel{3}}\right]_0^2$$

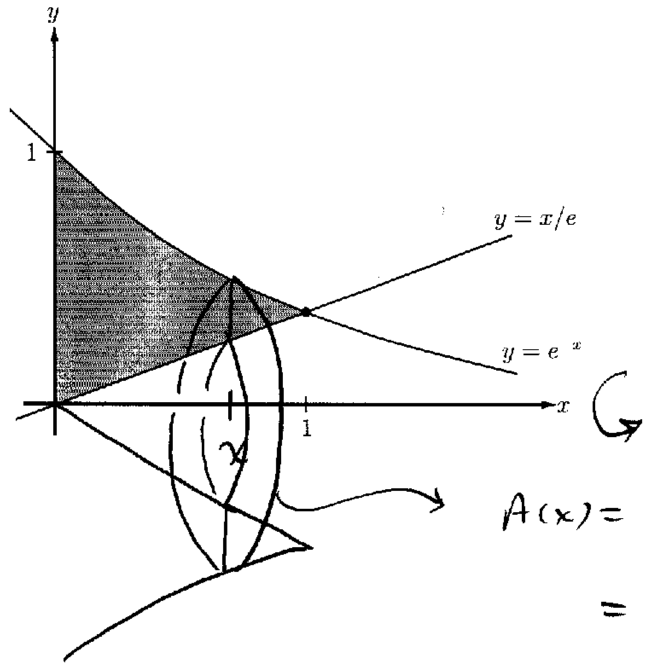
$$= -\frac{3}{8}\left[(2-y)^3\right]_0^2$$

$$= -\frac{3}{8}\left[0-8\right]$$

$$= \boxed{3}$$

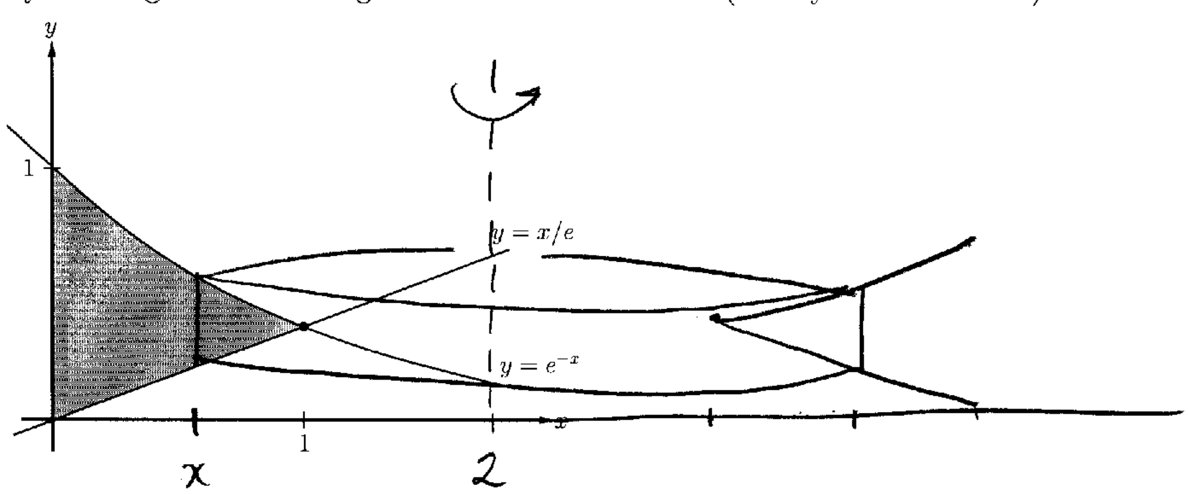
Question 5:

(a)[5 points] Set up but do not evaluate the integral representing the volume of the solid obtained by rotating the shaded region about the x-axis. (Use the disk method.)



$$A(x) = \pi (e^{-x})^2 - \pi \left(\frac{x}{e}\right)^2$$
$$= \pi \left[e^{-2x} - \frac{x^2}{e^2} \right]$$
$$\therefore V = \int_0^1 \pi \left[e^{-2x} - \frac{x^2}{e^2} \right] dx$$

(a)[5 points] Set up but do not evaluate the integral representing the volume of the solid obtained by rotating the shaded region about the line $x = 2$. (Use cylindrical shells.)



$$V = \int_0^1 2\pi (2-x)\left(e^{-x} - \frac{x}{e} \right) dx$$