

Question 1:

(a)[3 points] Let  $f(x) = \arcsin(\sinh(x))$ . Evaluate  $f'(0)$ .

$$f'(x) = \frac{1}{\sqrt{1 - \sinh^2(x)}} \cdot \cosh(x)$$

$$f'(0) = \frac{1}{\sqrt{1 - 0^2}} \cdot 1 = \boxed{1}$$

(b)[4 points] Evaluate

$$\lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{x \sin x} \sim \frac{0}{0}$$

$$\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{2x e^{x^2}}{\sin x + x \cos x} \sim \frac{0}{0}$$

$$\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{2e^{x^2} + (2x)^2 e^{x^2}}{\cos x + \cos x - x \sin x}$$

$$= \frac{2}{1+1}$$

$$= \boxed{1}$$

(c)[5 points] Evaluate

$$L = \lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}} \sim 1^\infty$$

$$= \lim_{x \rightarrow 0} e^{\frac{1}{x^2} \ln(\cos x)}$$

$$\lim_{x \rightarrow 0} \frac{\ln(\cos x)}{x^2} \sim \frac{0}{0}$$

$$\boxed{\therefore L = e^{-\frac{1}{2}}}$$

$$\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-\frac{\sin x}{\cos x}}{2x} \sim \frac{0}{0}$$

$$\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-\cos^2 x - \sin^2 x}{2 \cos^2 x} = \frac{-1}{2}$$

Question 2:

- (a)[4 points] A particle is moving with acceleration  $a(t) = t + \sin t$  m/s<sup>2</sup>. If velocity at time  $t = 0$  is  $v(0) = 2$  m/s and initial position is  $s(0) = 0$  m, determine  $s(t)$ , the position of the particle at time  $t$  = seconds.

$$v(t) = \frac{t^2}{2} - \cos t + C_1$$

$$v(0) = 2 \Rightarrow \frac{0^2}{2} - \cos(0) + C_1 = 2, \therefore C_1 = 3$$

$$\therefore v(t) = \frac{t^2}{2} - \cos(t) + 3$$

$$\therefore s(t) = \frac{t^3}{6} - \sin(t) + 3t + C_2$$

$$s(0) = 0 \Rightarrow \frac{0^3}{6} - \sin(0) + 3 \cdot 0 + C_2 = 0, \therefore C_2 = 0$$

$$\therefore \boxed{s(t) = \frac{t^3}{6} - \sin(t) + 3t}$$

- (b)[3 points] The average value of  $f(x) = x^3$  over the interval  $[0, a]$  is 16. Determine the value of  $a$ .

$$\frac{1}{a-0} \int_0^a x^3 dx = 16.$$

$$\frac{1}{a} \left[ \frac{x^4}{4} \right]_0^a = 16$$

$$\frac{1}{a} \cdot \frac{a^4 - 0^4}{4} = 16$$

$$a^3 = 64$$

$$\boxed{a = 4}$$

- (c)[3 points] Use the Fundamental Theorem of Calculus to determine  $f(x)$  if

$$\int_0^x f(t) dt = xe^{x^2} + 1.$$

$$\frac{d}{dx} \left[ \int_0^x f(t) dt \right] = \frac{d}{dx} [xe^{x^2} + 1]$$

$$f(x) = 1 \cdot e^{x^2} + x \cdot e^{x^2} \cdot (2x)$$

$$\boxed{f(x) = e^{x^2} + 2x^2 e^{x^2}}$$

Question 3:

(a)[3 points] Evaluate:

$$\begin{aligned} & \int \frac{2x^3 + 5\sqrt{x} - 3}{x} dx \\ &= 2 \int x^2 dx + 5 \int x^{-\frac{1}{2}} dx - 3 \int \frac{1}{x} dx \\ &= \frac{2x^3}{3} + \frac{5x^{\frac{1}{2}}}{\frac{1}{2}} - 3 \ln|x| + C \\ &= \boxed{\frac{2}{3}x^3 + 10x^{\frac{1}{2}} - 3 \ln|x| + C} \end{aligned}$$

(b)[3 points] Evaluate:

$$I = 2 \int \frac{\sin(\sqrt{x}+1)}{2\sqrt{x}} dx$$

$$\text{Let } u = \sqrt{x} + 1, \quad du = \frac{1}{2\sqrt{x}} dx$$

$$\therefore I = 2 \int \sin(u) du$$

$$= -2 \cos(u) + C$$

$$= \boxed{-2 \cos(\sqrt{x}+1) + C}$$

(c)[4 points] The population is growing at a rate of  $P'(t) = 200e^{t/5}$  individuals per year, where year  $t = 0$  corresponds to the present. If the current population size is  $P(0) = 1000$ , determine the population size at the end of five years.

$$\begin{aligned} P(5) - P(0) &= \int_0^5 P'(t) dt \\ \therefore P(5) &= 1000 + \int_0^5 200 e^{\frac{t}{5}} dt \\ &= 1000 + \frac{200}{5} \left[ e^{\frac{t}{5}} \right]_0^5 \\ &= 1000 + 1000 (e^1 - e^0) \\ &= 1000 (1 + e^1 - 1) = \boxed{1000e \text{ individuals.}} \end{aligned}$$

Question 4:

(a)[6 points] Evaluate:

$$\begin{aligned} I &= \int (x^2 + 1)e^{-x} dx \\ u &= x^2 + 1; \quad dv = e^{-x} dx \\ du &= 2x dx; \quad v = -e^{-x} \\ \therefore I &= -(x^2 + 1)e^{-x} + \int \underbrace{2x e^{-x} dx}_{u=2x; \quad dv=e^{-x} dx} \\ &\quad du = 2x dx; \quad v = -e^{-x} \end{aligned}$$

$$\therefore I = -(x^2 + 1)e^{-x} - 2x e^{-x} + \int 2e^{-x} dx$$

$$= \boxed{-(x^2 + 1)e^{-x} - 2x e^{-x} - 2e^{-x} + C}$$

(b)[6 points] Evaluate:

$$\begin{aligned} I &= \int \sec^6 x \tan^2 x dx \\ &= \int \sec^4 x \tan^2 x \sec^2 x dx \\ &= \int (1 + \tan^2 x)^2 \tan^2 x \sec^2 x dx \end{aligned}$$

$$u = \tan x, \quad du = \sec^2 x dx$$

$$\begin{aligned} \therefore I &= \int (1 + u^2)^2 u^2 du \\ &= \int u^2 + 2u^4 + u^6 du \\ &= \frac{u^3}{3} + \frac{2u^5}{5} + \frac{u^7}{7} + C \\ &= \boxed{\frac{\tan^3 x}{3} + \frac{2 \tan^5 x}{5} + \frac{\tan^7 x}{7} + C} \end{aligned}$$

Question 5 [8 points]: Evaluate:

$$I = \int x^3 \sqrt{4-x^2} dx$$

$$x = 2 \sin \theta$$

$$dx = 2 \cos \theta d\theta$$

$$\therefore I = \int 8 \sin^3 \theta \sqrt{4 - 4 \sin^2 \theta} 2 \cos \theta d\theta$$

$$= 32 \int \sin^3 \theta \cos^2 \theta d\theta$$

$$= 32 \int (1 - \cos^2 \theta) \cos^2 \theta \sin \theta d\theta$$

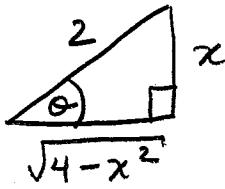
$$u = \cos \theta, du = -\sin \theta d\theta$$

$$\therefore I = -32 \int (1-u^2) u^2 du$$

$$= -32 \int u^2 - u^4 du$$

$$= -32 \left[ \frac{u^3}{3} - \frac{u^5}{5} \right] + C$$

$$= -32 \left[ \frac{\cos^3 \theta}{3} - \frac{\cos^5 \theta}{5} \right] + C$$



$$\therefore I = -32 \left[ \frac{1}{3} \left( \frac{\sqrt{4-x^2}}{2} \right)^3 - \frac{1}{5} \left( \frac{\sqrt{4-x^2}}{2} \right)^5 \right] + C$$

$$= \boxed{-\frac{4}{3} (4-x^2)^{\frac{3}{2}} + \frac{1}{5} (4-x^2)^{\frac{5}{2}} + C}$$

Question 6 [8 points]: Evaluate:

$$I = \int_0^1 \frac{4x^2 + x + 3}{(x+1)(x^2+1)} dx$$

$$\begin{aligned} \frac{4x^2 + x + 3}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \\ &= \frac{Ax^2 + A + Bx^2 + Cx + Bx + C}{(x+1)(x^2+1)} \\ &= \frac{(A+B)x^2 + (C+B)x + A+C}{(x+1)(x^2+1)} \end{aligned}$$

$$\therefore A+B = 4 \Rightarrow B = 4-A$$

$$C+B = 1 \Rightarrow C + 4-A = 1 \Rightarrow C = A-3$$

$$A+C = 3 \Rightarrow A + (A-3) = 3 \Rightarrow 2A = 6 \Rightarrow A = 3$$

$$\therefore C = 3-3 = 0$$

$$\therefore B = 4-3 = 1$$

$$\therefore I = \int_0^1 \frac{3}{x+1} + \frac{x}{x^2+1} dx$$

$$= 3 \int_0^1 \frac{1}{x+1} dx + \frac{1}{2} \int_0^1 \frac{2x}{x^2+1} dx$$

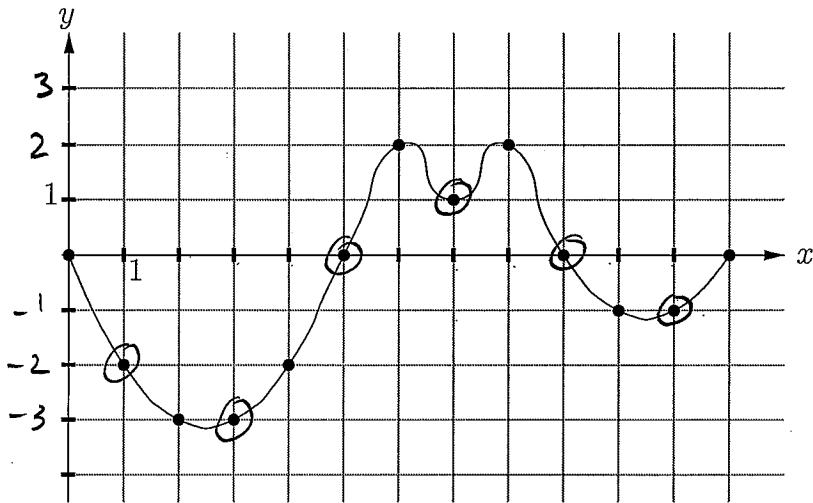
$$= 3 \left[ \ln|x+1| \right]_0^1 + \frac{1}{2} \left[ \ln|x^2+1| \right]_0^1$$

$$= 3 \left( \ln(2) - \cancel{\ln(1)}^0 \right) + \frac{1}{2} \left( \ln(z) - \cancel{\ln(1)}^0 \right)$$

$$= \boxed{\frac{7}{2} \ln 2}$$

Question 7:

- (a)[4 points] Use the graph of  $y = f(x)$  below to approximate  $\int_0^{12} f(x) dx$  using  $M_6$ , the midpoint rule on six subintervals.



$$\Delta x = \frac{12}{6} = 2$$

$$\therefore \int_0^{12} f(x) dx \approx M_6 = 2 \left[ (-2) + (-3) + 0 + 1 + 0 + (-1) \right]$$

$$= \boxed{-10}$$

- (b)[4 points] If Simpson's Rule is used to approximate  $\int_0^{180} \left( \frac{x^4 + x}{24} \right) dx$ , how many subintervals are required to guarantee an error of at most 1? Recall, the error in using Simpson's rule to approximate  $\int_a^b f(x) dx$  is at most  $\frac{K(b-a)^5}{180n^4}$ , where  $|f^{(4)}(x)| \leq K$  on  $[a, b]$ .

$$f(x) = \frac{x^4 + x}{24} \quad \therefore \quad \frac{K(b-a)^5}{180n^4} \leq 1$$

$$f'(x) = \frac{4x^3}{24} \quad \frac{1 \cdot (180-0)^5}{180n^4} \leq 1$$

$$f''(x) = \frac{12x^2}{24} \quad 180^4 \leq n^4$$

$$f'''(x) = \frac{24x}{24} \quad 180 \leq n$$

$$f^{(iv)}(x) = 1 \quad \therefore \quad n \geq 180$$

$$\therefore K = 1$$

Question 8:

- (a)[3 points] Use the comparison theorem to determine if the integral converges or diverges (do not attempt to evaluate the integral):

$$\int_1^\infty \frac{e^{-x}}{x^4+1} dx$$

$$0 \leq \frac{e^{-x}}{x^4+1} \leq \frac{1}{x^4} \quad \text{on } [1, \infty),$$

Since  $\int_1^\infty \frac{1}{x^4} dx$  converges, so

must  $\int_1^\infty \frac{e^{-x}}{x^4+1} dx.$

- (b)[5 points] Evaluate the improper integral:

$$\int_0^1 \frac{e^x}{\sqrt{e^x-1}} dx$$

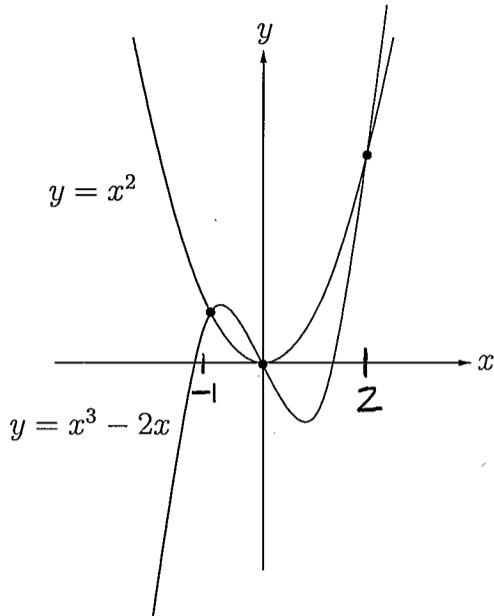
For  $I = \int \frac{e^x}{\sqrt{e^x-1}} dx$ , let  $u = e^x - 1$ ,  $du = e^x dx$ .

$$\therefore I = \int u^{-\frac{1}{2}} du = 2u^{\frac{1}{2}} + C = 2\sqrt{e^x-1} + C$$

$$\begin{aligned} \therefore \int_0^1 \frac{e^x}{\sqrt{e^x-1}} dx &= \lim_{t \rightarrow 0^+} \int_t^1 \frac{e^x}{\sqrt{e^x-1}} dx \\ &= \lim_{t \rightarrow 0^+} \left[ 2\sqrt{e^x-1} \right]_t^1 \\ &= \lim_{t \rightarrow 0^+} \left( 2\sqrt{e-1} - 2\sqrt{e^t-1} \right) \\ &= \boxed{2\sqrt{e-1}} \end{aligned}$$

Question 9:

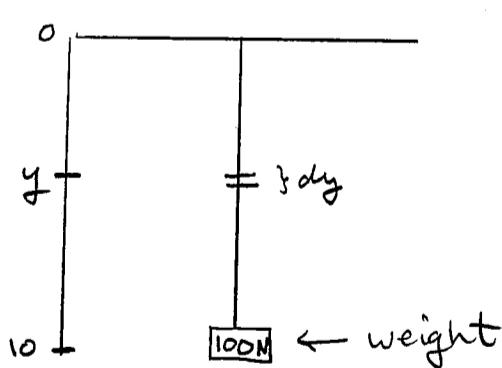
- (a)[5 points] Determine the area of the region bounded between the curves  $y = x^2$  and  $y = x^3 - 2x$ .



$$\begin{aligned} x^2 &= x^3 - 2x \\ x^3 - x^2 - 2x &= 0 \\ x(x^2 - x - 2) &= 0 \\ x(x+1)(x-2) &= 0 \\ \therefore x = 0, x = -1, x = 2 \end{aligned}$$

$$\begin{aligned} \therefore A &= \int_{-1}^0 (x^3 - 2x - x^2) dx + \int_0^2 (x^2 - x^3 + 2x) dx \\ &= \left[ \frac{x^4}{4} - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^0 + \left[ \frac{x^3}{3} - \frac{x^4}{4} + x^2 \right]_0^2 \\ &= \left( \frac{0^4}{4} - 0^2 - \frac{0^3}{3} \right) - \left( \frac{1}{4} - 1 + \frac{1}{3} \right) + \left( \frac{8}{3} - \frac{16}{4} + 4 \right) - \left( \frac{0^3}{3} - \frac{0^4}{4} + 0^2 \right) \\ &= \frac{5}{12} + \frac{8}{3} = \frac{5+32}{12} = \boxed{\frac{37}{12}} \end{aligned}$$

- (b)[5 points] A 10 m chain with a 100 N weight at the end hangs over the side of a bridge. The weight is initially 20 m above the ground, and one metre of chain weighs 10 N. A person pulls the chain and weight up onto the bridge deck. How much work is done?



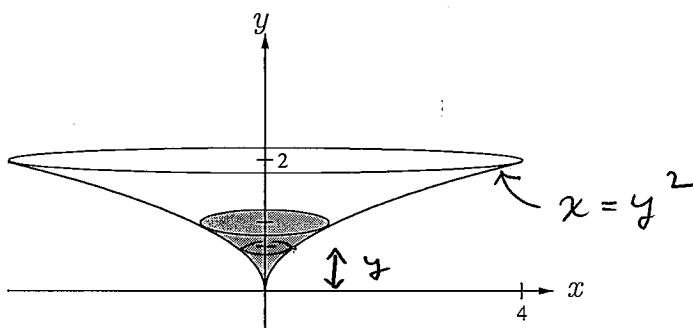
$$W_{\text{weight}} = F \cdot d = (100 \text{ N})(10 \text{ m}) = 1000 \text{ J.}$$

$$\begin{aligned} W_{\text{chain}} &= \int_0^{10} (y \text{ m}) (10 \frac{\text{N}}{\text{m}}) (dy \text{ m}) \\ &= 10 \int_0^{10} y dy \\ &= \frac{10}{2} [y^2]_0^{10} \\ &= 500 \text{ J} \end{aligned}$$

$$\therefore \text{Total work} = 1000 + 500 = \boxed{1500 \text{ J}}$$

Question 10

- (a) [7 points] A vessel is formed by rotating the curve  $y = \sqrt{x}$  about the  $y$ -axis as shown below. The vessel has a top radius of 4 m and a depth of 2 m. If the vessel is initially filled with water to a depth of 1 m, how much work is required empty the vessel by pumping the water up and over the top rim? Recall that the density of water is  $\rho = 1000 \text{ kg/m}^3$  and acceleration due to gravity is  $g = 9.8 \text{ m/s}^2$ . (You may leave your final answer as a product of fractions.)



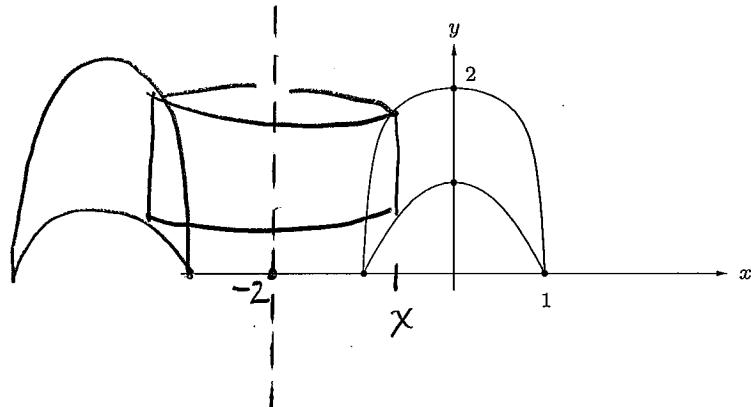
$$\begin{aligned}
 W &= \int_0^1 9.8(1000) \pi (y^2)^2 (2-y) dy \\
 &= 9800 \pi \int_0^1 2y^4 - y^5 dy \\
 &= 9800 \pi \left( \frac{2}{5} [y^5]_0^1 - \frac{1}{6} [y^6]_0^1 \right) \\
 &= 9800 \pi \left( \frac{2}{5} - \frac{1}{6} \right) \\
 &= 9800 \pi \left( \frac{12-5}{30} \right) \\
 &= 9800 \pi \left( \frac{7}{30} \right) \\
 &= \boxed{980 \pi \left( \frac{7}{30} \right) \text{ J}}
 \end{aligned}$$

- (b) [3 points] Determine the total volume of the vessel described in part (a). (Disks would be best here.)

$$\begin{aligned}
 V &= \int_0^2 \pi (y^2)^2 dy \\
 &= \frac{\pi}{5} [y^5]_0^2 \\
 &= \boxed{\frac{32\pi}{5}}
 \end{aligned}$$

Question 11:

- (a)[3 points] The region bounded between the curves  $y = 1 - x^2$  and  $y = 2(1 - x^4)$  is rotated about the line  $x = -2$ . Set up the integral representing the volume of the resulting solid. DO NOT EVALUATE THE INTEGRAL. (Cylindrical shells would be best here.)

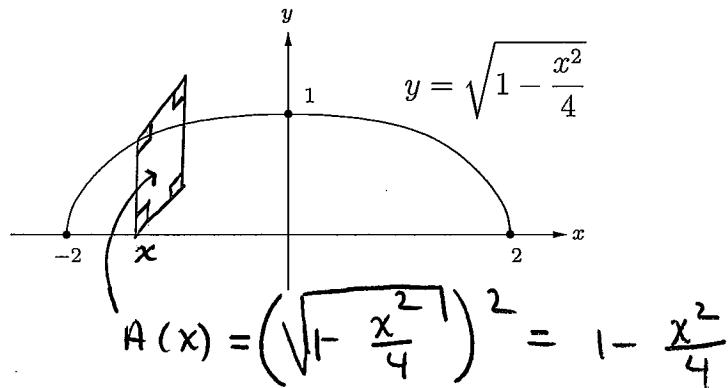


$$V = \int_{-1}^1 2\pi (x+2) (2(1-x^4) - (1-x^2)) dx$$

- (b)[3 points] Referring to the diagram for part (a), the same bounded region is rotated about the horizontal line  $y = 2$ . Set up the integral representing the volume of the resulting solid. DO NOT EVALUATE THE INTEGRAL. (Disks would be best here.)

$$V = \int_{-1}^1 \pi \left[ (2-(1-x^2))^2 - (2-2(1-x^4))^2 \right] dx$$

- (c)[4 points] The base (flat bottom surface) of the solid  $S$  is the region between the curve  $y = \sqrt{1 - \frac{x^2}{4}}$  and the  $x$ -axis. Cross-sections perpendicular to the  $x$ -axis are squares. Determine the volume of  $S$ .



$$\begin{aligned} \therefore V &= \int_{-2}^2 1 - \frac{x^2}{4} dx = 2 \int_0^2 1 - \frac{1}{4} x^2 dx \\ &= 2 \left[ x - \frac{1}{12} x^3 \right]_0^2 \\ &= 2 \left[ 2 - \frac{8}{12} \right] = \boxed{\frac{8}{3}} \end{aligned}$$

Question 12:

(a)[4 points] Solve the differential equation:

$$\frac{dy}{dx} = \frac{y \cos x}{1 + y^2} \quad y(0) = 1$$

$$\int \frac{1+y^2}{y} dy = \int \cos x dx$$

$$\int \frac{1}{y} + y^2 dy = \int \cos x dx$$

$$\ln|y| + \frac{y^2}{2} = \sin x + C$$

$$y(0) = 1 \Rightarrow \ln|1| + \frac{1^2}{2} = \sin(0) + C$$

$$\Rightarrow C = \frac{1}{2}$$

$$\therefore \boxed{\ln|y| + \frac{y^2}{2} = \sin x + \frac{1}{2}}$$

(b)[6 points] A tank contains 1000 L of pure water. Salt water with a concentration of 0.1 kg/L of salt enters the tank at a rate of 10 L per minute, while water is pumped out of the bottom of the tank at the same rate. The water in the tank is kept thoroughly mixed. The mass  $A(t)$  of dissolved salt in the tank at time  $t$  is modeled by the differential equation

$$\frac{dA}{dt} = 1 - \frac{1}{100}A \quad A(0) = 0$$

Solve this differential equation for  $A(t)$ .

$$\left. \begin{array}{l} \int \frac{1}{1 - \frac{1}{100}A} dA = \int dt \\ -100 \ln \left| 1 - \frac{1}{100}A \right| = t + C_1 \\ \ln \left| 1 - \frac{1}{100}A \right| = -\frac{1}{100}t + C_2 \\ \left| 1 - \frac{1}{100}A \right| = C_3 e^{-\frac{1}{100}t} \\ 1 - \frac{1}{100}A = C_4 e^{-\frac{1}{100}t} \\ A = 100 - C_5 e^{-\frac{1}{100}t} \end{array} \right\} \begin{array}{l} A(0) = 0, \quad \text{so} \\ 0 = 100 - C_5 e^0 \\ \therefore C_5 = 100, \\ \therefore A(t) = 100 - 100 e^{-\frac{1}{100}t} \end{array}$$

Question 13:

- (a)[4 points] Use the definition to construct  $T_2(x)$ , the Maclaurin series of degree 2, for the function  $f(x) = \sin(x^2 + x)$

$$f(x) = \sin(x^2 + x) ; f(0) = \sin(0^2 + 0) = 0$$

$$f'(x) = \cos(x^2 + x)(2x+1) ; f'(0) = \cos(0^2 + 0)(2 \cdot 0 + 1) = 1$$

$$f''(x) = -\sin(x^2 + x)(2x+1)^2 + \cos(x^2 + x) \cdot 2 ; f''(0) = 0 + 2 = 2$$

$$\therefore T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

$$= 0 + 1 \cdot x + 2 \cdot \frac{x^2}{2}$$

$$= \boxed{x + x^2}$$

- (b)[4 points] Use Maclaurin series (not L'Hospital's Rule) to evaluate the limit

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots\right) - 1}{x \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)}$$

$$= \lim_{x \rightarrow 0} \frac{-\frac{x^2}{2} + \frac{x^4}{4!} - \dots}{x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \dots}$$

$$= \lim_{x \rightarrow 0} \frac{-\frac{1}{2} + \frac{x^2}{4!} - \dots}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots}$$

$$= \boxed{-\frac{1}{2}}$$