Question 1:

(a)[7 points] Let $f(x) = \dfrac{1}{x+1}$. Evaluate and simplify the difference quotient

$$\frac{f(x+h)-f(x)}{h}.$$

$$= \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}$$

$$= \frac{1}{h}\left[\frac{\cancel{x}\cancel{+1}\cancel{-x}-h\cancel{-1}}{(x+1)(x+h+1)}\right]$$

$$= \frac{1}{\cancel{h}}\left[\frac{-\cancel{h}}{(x+1)(x+h+1)}\right]$$

$$= \boxed{\frac{-1}{(x+1)(x+h+1)}}$$

(b)[3 points] Find the domain of the function $f(x) = \dfrac{1}{\sqrt{x^2-4}}$.

Must have $x^2-4>0$

$\therefore\ x^2>4$

$\therefore\ x>2$ or $x<-2$

$\therefore\ \boxed{(-\infty,-2)\cup(2,\infty)}$

Question 2:

(a)[3 points] Let $f(x) = \sqrt{x+3}$ and $g(x) = x^2 - 3$. Find $g \circ f$ and state the domain.

$$(g \circ f)(x) = g(f(x))$$
$$= (\sqrt{x+3})^2 - 3$$
$$= \boxed{x}$$

must have $x + 3 \geq 0$,
i.e. $x \geq -3$

$\therefore$ $\boxed{\text{domain : } [-3, \infty)}$

(b)[4 points] Again let $f(x) = \sqrt{x+3}$ and $g(x) = x^2 - 3$. Find $\frac{f}{g}$ and state the domain.

$$\left(\frac{f}{g}\right)(x) = \boxed{\frac{\sqrt{x+3}}{x^2-3}}$$

must have
$x + 3 \geq 0$, $x^2 - 3 \neq 0$
i.e $x \geq -3$, $x \neq \pm\sqrt{3}$

$\boxed{\therefore \text{ domain is } [-3, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty).}$

(c)[3 points] Let $F(x) = 1 + \sqrt{\sin x}$. Find functions f and g so that $F = f \circ g$.

Let $g(x) = \sin x$
$f(x) = 1 + \sqrt{x}$

Question 3:

(a)[5 points] Evaluate the limit if it exists:

$$\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4}.$$

$$= \lim_{x \to 4} \frac{x\cancel{(x-4)}}{(x+1)\cancel{(x-4)}}$$

$$= \boxed{\frac{4}{5}}$$

(b)[5 points] Evaluate the limit if it exists:

$$\lim_{h\to 0} \frac{\frac{1}{2+h}-\frac{1}{2}}{h}.$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{2-2-h}{(2+h)2}\right]$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left[\frac{-\cancel{h}}{(2+h)2}\right]$$

$$= \boxed{\frac{-1}{4}}$$

Question 4:

(a)[5 points] Evaluate the limit if it exists:

$$\lim_{x \to 0} x^2 \cos\left(\frac{1+x^2}{x^2-x}\right).$$

$$-x^2 \leq x^2 \cos\left(\frac{1+x^2}{x^2-x}\right) \leq x^2$$

Since $\lim\limits_{x\to 0} -x^2 = \lim\limits_{x \to 0} x^2 = 0$,

by the Squeeze Theorem

$$\lim_{x\to 0} x^2 \cos\left(\frac{1+x^2}{x^2-x}\right) = \boxed{0}$$

(b)[5 points] Evaluate the limit if it exists:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta + \tan\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta + \frac{\sin\theta}{\cos\theta}} \begin{matrix} \div \theta \\ \div \theta \end{matrix}$$

$$= \lim_{\theta \to 0} \frac{\left(\frac{\sin\theta}{\theta}\right) \} \to 1}{1 + \underbrace{\left(\frac{\sin\theta}{\theta}\right)}_{\to 1} \underbrace{\left(\frac{1}{\cos\theta}\right)}_{\to 1}}$$

$$= \boxed{\frac{1}{2}}$$

Question 5:

(a)[5 points] Let

$$g(x) = \begin{cases} x^4 - cx^2 & \text{if } x < 2 \\ c^2x + 18 & \text{if } x \geq 2 \end{cases}$$

Find the constant c that makes g continuous at all real numbers.

Must have $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x) = g(2)$

i.e $\lim_{x\to 2^-} x^4 - cx^2 = \lim_{x\to 2^+} c^2x + 18 = c^2(2) + 18$

$$\therefore \quad 16 - 4c = 2c^2 + 18$$

$$2c^2 + 4c + 2 = 0$$

$$c^2 + 2c + 1 = 0$$

$$(c+1)^2 = 0$$

$$\boxed{c = -1}$$

(b)[5 points] Evaluate the limit if it exists:

$$\lim_{x\to\infty}\left(\sqrt{x^2-3x} - \sqrt{x^2-5x}\right) \frac{\left(\sqrt{x^2-3x} + \sqrt{x^2-5x}\right)}{\sqrt{x^2-3x} + \sqrt{x^2-5x}}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2} - 3x - \cancel{x^2} + 5x}{\sqrt{x^2-3x} + \sqrt{x^2-5x}}$$

$$= \lim_{x\to\infty} \frac{2\cancel{x}}{\cancel{x}\sqrt{1-\frac{3}{x}} + \cancel{x}\sqrt{1-\frac{5}{x}}}$$

$$= \frac{2}{1+1}$$

$$= \boxed{1}$$