1 Real Numbers

In this course we work mainly with **real numbers**: the number system which includes all numbers we have encountered in our mathematical training to date. Our first exposure to numbers usually involves the

whole numbers: $0, 1, 2, 3, \ldots,$

followed by the

integers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots.$

After that we learn to do math with the

rational numbers: all possible fractions we can form with the integers.

Here's another way to describe the rational numbers:

rational numbers: all a/b where a and b are integers and b is not zero.

Here are some examples of rational numbers:

$$\frac{1}{2}, \frac{-7}{3}, \frac{5}{5}$$

Notice that, for example, -7 can be written as $-7/1$, so the the rational numbers include the integers.

There are some numbers that cannot be written as fractions of integers and so are not rational numbers. The most famous such number is π, but there are others, like $\sqrt{2}$ for example. These are called the **irrational numbers**.

The rational and irrational numbers together form the **real numbers**. In summary, the real numbers include the rational and irrational numbers. The rational numbers include the integers, which in turn include the whole numbers.

2 Exponent Laws

Recall the notation for exponents: if a is any real number and n is any positive integer, then

$$a^n \qquad \text{means} \qquad \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}$$

So, for example,

$$2^4 = (2)(2)(2)(2) = 16 \ .$$

We also define

$$a^{-n} = \frac{1}{a^n}$$

and

$$a^0 = 1 \qquad \text{as long as } a \neq 0$$

With this notation we now state the **exponent laws**. Let m, n, a and b be any real numbers.Then

1. $a^m a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^n = a^n b^n$
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
6. $\left(\dfrac{a}{b}\right)^{-n} = \dfrac{b^n}{a^n}$

(Each of these rules is valid as long as both sides of the equality are defined.)

Example 1: Simplify

$$\left(\frac{x^{-3}}{x^{-5}}\right)^{-2}$$

Solution:

$$\begin{aligned}\left(\frac{x^{-3}}{x^{-5}}\right)^{-2} &= \left(\frac{(x^{-3})^{-2}}{(x^{-5})^{-2}}\right)\\ &= \frac{x^6}{x^{10}}\\ &= x^{6-10}\\ &= x^{-4}\end{aligned}$$

□

Example 2: Simplify

$$(2^4x^2y^{-2}z^3)(4^2x^{-2}y^3z^5)$$

Solution:

$$\begin{aligned}(2^4x^2y^{-2}z^3)(4^2x^{-2}y^3z^5) &= (2^4)(4^2)(x^2x^{-2})(y^{-2}y^3)(z^3z^5)\\ &= (2^4)((2^2)^2)x^0y^1z^8\\ &= (2^4)(2^4)yz^8\\ &= 2^8yz^8\end{aligned}$$

□

3 Expanding and Factoring

Expanding and expression simply means "multiplying out". Factoring (the harder of the two) is the reverse: write the expression as a product of factors which cannot be factored further.

3.1 Expanding

Expanding usually involves multiplying out terms, some of which are in parentheses. The basic principle is: multiply each term in the first set of parentheses with each term in the second.

Example 3: Expand and simplify

$$ac(ab + c^2)$$

Solution:

$$\begin{aligned} ac(ab + c^2) &= (ac)(ab + c^2) \\ &= (ac)(ab) + (ac)(c^2) \\ &= a^2bc + ac^3 \end{aligned}$$

□

A very common expansion is that involving the product of two binomials, and is typically recalled using the FOIL rule:

$$(a + b)(c + d) = ac + ad + bc + bd$$

Example 4: Expand and simplify

$$(4 - x^2)(x - y^3)$$

Solution:

$$(4 - x^2)(x - y^3) = 4x - 4y^3 - x^3 + x^2y^3$$

□

3.2 Factoring

Factoring is expanding in reverse. The simplest case is when each term contains a common factor, and you should **always** check for this first:

Example 5: Factor completely

$$a^2b^3 - 3a^3b^2$$

Solution:

$$a^2b^3 - 3a^3b^2 = a^2b^2(b - 3a)$$

□

Factoring trinomials of the form $x^2 + bx + c$ is very common and follows a basic strategy:

$$x^2 + bx + c = (x + u)(x + v)$$

where $u + v = b$ and $uv = c$. That is, to factor $x^2 + bx + c$, find two numbers which add to give b and multiply to give c.

Example 6: Factor completely

$$x^2 + 3x - 10$$

Solution: We need two numbers which add to 3 and multiply to give -10. The two numbers -2 and 5 work, so

$$x^2 + 3x - 10 = (x - 2)(x + 5)$$

□

Another common pattern which arises in factoring problems is a **difference of squares**:

$$x^2 - y^2 = (x - y)(x + y)$$

Example 7: Factor completely

$$64a^4 - 4a^2$$

Solution:

$$\begin{aligned} 64a^4 - 4a^2 &= 4a^2(16a^2 - 1) \\ &= 4a^2(4a - 1)(4a + 1) \end{aligned}$$

□

4 Simplifying Fractions

We'll deal with fraction algebra in a moment, but for now we wish to make clear a very important point which is often ignored and leads to errors: when reducing fractions by canceling factors, the factor being canceled must be common to **every** term in both the numerator and denominator.

Example 8: Simplify

$$\frac{4x + x^3}{x^2}$$

Solution: x is common to all terms in both numerator and denominator, and to demonstrate that you should be able to factor it out of every term:

$$\begin{aligned}\frac{4x + x^3}{x^2} &= \frac{(x)(4 + x^2)}{(x)(x)} \\ &= \frac{\cancel{(x)}(4 + x^2)}{\cancel{(x)}(x)} \\ &= \frac{4 + x^2}{x}\end{aligned}$$

□

In the previous example you may be tempted to cancel the remaining x's BUT DON'T: x is not common to all terms in $\frac{4 + x^2}{x}$ so may NOT be canceled.

5 Algebra with Fractions

Algebra involving fractions can sometimes look imposing, but it is just a matter of applying the same rules used for ordinary fractions (that is, fractions involving only numbers), and remembering the simplification rule of the previous section.

5.1 Addition and Subtraction

First we tackle addition and subtraction. To add (or subtract) two fractions,the denominators must be the same. Recall the procedure when the fractions involve only numbers:

1. Find a common denominator (the least common multiple of the denominators is generally best).
2. Adjust the numerators to reflect the new denominators and maintain equality.
3. Add (or subtract) the new numerators, putting your result over the common denominator.
4. Reduce your answer by canceling common factors, if any.

Example 9: Simplify

$$\frac{7}{6} - \frac{3}{10}$$

Solution: The least common multiple of 6 and 10 is 30, our common denominator. Now adjust the numerators: for the first fraction

$$\frac{7}{6} = \frac{?}{30} .$$

The denominator 6 is multiplied by 5 to get 30, so the same must be done to the numerator:

$$\frac{7 \times 5}{6 \times 5} = \frac{35}{30} .$$

For the second fraction the denominator 10 must be multiplied by 3:

$$\frac{3}{10} = \frac{?}{30}$$

$$\frac{3 \times 3}{10 \times 3} = \frac{9}{30} .$$

So the calculation is

$$\frac{7}{6} - \frac{3}{10} = \frac{35}{30} - \frac{9}{30} = \frac{26}{30} .$$

Finally, cancel any common factors to get the final answer:

$$\frac{26}{30} = \frac{\cancel{(2)}(13)}{\cancel{(2)}(15)} = \frac{13}{15} .$$

□

Now apply the same procedure when variables are added to the problem.

Example 10: Simplify

$$\frac{7}{6ab} - \frac{3a}{10b^2}$$

Solution: To find the common denominator, first find the least common multiple of 6 and 10, again 30, then find an expression into which the variable factors in the denominators divide exactly. Notice that both ab and b^2 divide into ab^2. So our common denominator is $30ab^2$. Now adjust numerators as before:

$$\frac{7}{6ab} = \frac{?}{30ab^2}$$

$$\frac{(7)(5b)}{(6ab)(5b)} = \frac{35b}{30ab^2} ,$$

and

$$\frac{3a}{10b^2} = \frac{?}{30ab^2}$$

$$\frac{(3a)(3a)}{(10b^2)(3a)} = \frac{9a^2}{30ab^2} .$$

So the calculation is

$$\frac{7}{6ab} - \frac{3a}{10b^2} = \frac{35b}{30ab^2} - \frac{9a^2}{30ab^2} = \frac{35b - 9a^2}{30ab^2} .$$

No factor is common to all terms of the numerator and denominator in this last fraction, and so we can't simplify further. □

5.2 Multiplication and Division

5.2.1 Multiplication

Multiplication is the easiest of the operations with fractions: simply multiply numerators and denominators:

Example 11: Simplify

$$\left(\frac{7}{6ab}\right)\left(\frac{3a}{10b^2}\right)$$

Solution:

$$\begin{aligned}\left(\frac{7}{6ab}\right)\left(\frac{3a}{10b^2}\right) &= \frac{(7)(3a)}{(6ab)(10b^2)}\\ &= \frac{21a}{60ab^3}\\ &= \frac{7}{20b^3} \qquad \text{canceling } 3a \text{ from numerator and denominator .}\end{aligned}$$

□

5.2.2 Division

Division is almost as easy: one fraction divided by another is the same as the first fraction multiplied by the reciprocal of the second:

Example 12: Simplify

$$\frac{\dfrac{xy}{w}}{\left(\dfrac{xy-2x}{w}\right)}$$

Solution:

$$\begin{aligned}\frac{\dfrac{xy}{w}}{\left(\dfrac{xy-2x}{w}\right)} &= \frac{xy}{w}\frac{w}{xy-2x}\\ &= \frac{wxy}{w(xy-2x)}\\ &= \frac{wxy}{wx(y-2)}\\ &= \frac{y}{y-2} \qquad \text{canceling } wx \text{ from numerator and denominator .}\end{aligned}$$

□

6 Solving Linear Equations

Equations like, for example, $9x - 2 = 0$, are called **linear equations** in the variable x. More generally, a linear equation is an equation which can be expressed in the form

$$ax + b = 0$$

where a and b are real numbers and $a \neq 0$. They are called linear equations because expressions of the form $ax + b$ represent (straight) lines when graphed; we'll see this more later. Notice that in a linear equation, the variable x occurs to the first power; there are no x^2, x^3, $\sqrt{x}$, etc.

To **solve a linear equation** is to find the value of x which satisfies the equation. To solve a linear equation we isolate the unknown x by performing the same mathematically valid operations to both sides of the equation.

Example 13: Solve for x

$$9x - 2 = 0$$

Solution:

$$\begin{aligned} 9x - 2 &= 0 \\ 9x - 2 + 2 &= 0 + 2 \qquad \text{(adding 2 to both sides)} \\ 9x &= 2 \\ \frac{9x}{9} &= \frac{2}{9} \qquad \text{(dividing both sides by 9)} \\ x &= \frac{2}{9} \end{aligned}$$

A quick check (which you should always do!) shows that $x = 2/9$ is indeed a solution:

$$9\left(\frac{2}{9}\right) - 2 = 2 - 2 = 0\ .$$

□

Some more complicated equations can be linear equations in disguise. Furthermore, the variable need not be x. For example:

Example 14: Solve for w

$$\frac{1}{3}w - 2 = \frac{1}{2}w + 5$$

Solution: the strategy is to get all the terms with w on one side, and all the non-w terms on the other:

$$\begin{aligned}
\frac{1}{3}w - 2 &= \frac{1}{2}w + 5 \\
\frac{1}{3}w - \frac{1}{2}w - 2 &= 5 \\
\frac{1}{3}w - \frac{1}{2}w &= 5 + 2 \\
\left(\frac{1}{3} - \frac{1}{2}\right) w &= 7 \\
\left(\frac{2-3}{6}\right) w &= 7 \qquad \text{(notice the slightly different way of subtracting fractions here)} \\
\left(\frac{-1}{6}\right) w &= 7 \\
w &= 7\left(\frac{-6}{1}\right) \\
w &= -42 \ .
\end{aligned}$$

□

7 Solving Quadratic Equations

A **quadratic equation** is an equation which can be expressed in the form

$$ax^2 + bx + c = 0$$

where $a \neq 0$. To solve a quadratic equation is to find **all** values of x which satisfy the equation. What is new here is that the equation can have as many as two real number solutions, or possibly no (real) solutions at all.

7.1 Solving by Factoring

The first approach to solving a quadratic equation is by factoring:

Example 15: Solve for x

$$x^2 - 6x - 7 = 0$$

Solution: This expression looks easy to factor, so do it:

$$\begin{aligned}
x^2 - 6x - 7 &= 0 \\
(x-7)(x+1) &= 0
\end{aligned}$$

Now apply the fundamental principal: if two factors multiply to zero, then one of those factors (or both) must be zero (this is called the **zero factor property**). So either

$$(x-7)=0 \quad \text{or} \quad (x+1)=0\ ,$$
$$\text{i.e.}\quad x=7 \quad \text{or} \quad x=-1$$

□

7.2 Solving Using the Quadratic Formula

Some quadratic equations are not so easy to handle by factoring, but fortunately there is a handy tool for solving such equations: the **quadratic formula**. The quadratic equation written in the form

$$ax^2+bx+c=0$$

where $a \neq 0$ has solutions (or roots) given by

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\ .}$$

You should commit the quadratic formula to memory. The $\pm$ in the formula is just shorthand for expressing both solutions in one. The two solutions written separately are

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad x=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

The term b^2-4ac under the square root is called the *discriminant* and can be used to determine the number of solutions of the quadratic equation as follow:

- If $b^2-4ac>0$ there are two distinct real roots;
- If $b^2-4ac=0$ there is only one real root;
- If $b^2-4ac<0$ there are no real roots.

Here's an example of the quadratic formula in action:

Example 16: Solve for x

$$4x+5=3x^2$$

Solution: First, put the equation into the standard form $ax^2+bx+c=0$:

$$4x+5=3x^2$$
$$-3x^2+4x+5=0$$
$$3x^2-4x-5=0$$

Notice how we multiplied through by -1 in the last line above to eliminate the minus sign in the first term: not necessary but a bit cleaner. This doesn't look so easy to factor, so use the quadratic formula. Reading off the a, b and c from our equation we have

$$a = 3, \qquad b = -4, \qquad c = -5$$

Now for the formula:

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-5)}}{2(3)} \\
&= \frac{4 \pm \sqrt{16 + 60}}{6} \\
&= \frac{4 \pm \sqrt{76}}{6} \\
&= \frac{4 \pm \sqrt{(4)(19)}}{6} \\
&= \frac{4 \pm 2\sqrt{19}}{6} \\
&= \frac{2 \pm \sqrt{19}}{3} \quad \text{after canceling the 2 common to all terms}
\end{aligned}$$

So our two solutions are

$$x = \frac{2 + \sqrt{19}}{3} \text{ and } \frac{2 - \sqrt{19}}{3} .$$

□

Another example, this time resulting in no solutions:

Example 17: Solve for x

$$x^2 - \frac{2}{3}x = -1$$

Solution: Once again, as a first step put the equation into the standard form $ax^2 + bx + c = 0$:

$$\begin{aligned}
x^2 - \frac{2}{3}x &= -1 \\
x^2 - \frac{2}{3}x + 1 &= 0 \\
3x^2 - 2x + 3 &= 0
\end{aligned}$$

In the last line we multiplied through by 3 to clear the fractions in the equation. Multiplying both sides of an equation by a non-zero number (a *constant*) will not alter the solutions. Once again, factoring looks tough, so use the quadratic equation with

$$a = 3, \qquad b = -2, \qquad c = 3 \ :$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(3)}}{2(3)} \\ &= \frac{4 \pm \sqrt{-32}}{6} \end{aligned}$$

Since the discriminant (the term under the square root) is negative we can stop and conclude there are no real roots. □

When solving quadratic equations (or any equations for that matter), avoid the temptation of canceling variables appearing on both sides of the equal sign. For example, to solve

$$x^2 = x$$

you may be tempted to cancel the x on both sides, like so:

$$\begin{aligned} x^2 &= x \\ \cancel{x}^{\cancel{2}} &= \cancel{x} \\ x &= 1 \end{aligned}$$

but this is **WRONG!!!**. We've just lost one of our solutions! The proper way to solve in this case is by factoring:

$$\begin{aligned} x^2 &= x \\ x^2 - x &= 0 \\ x(x-1) &= 0 \end{aligned}$$

so $x = 0$ or $x = 1$.

8 Problems

1. Simplify $(3a^3b^4)^5(2ab^3c^7)^3$.

ans: $1944a^{18}b^{29}c^{21}$

2. Expand and simplify $5[9x - (3x + 2/5)]$.

ans: $30x - 2$

3. Expand $(\sqrt{u^2+v^2} - u)(\sqrt{u^2+v^2} + u)$.

ans: v^2

4. Simplify using only positive exponents: $\dfrac{x^{-3}y^{-2}}{x^2y^{-7}}$.

ans: $\dfrac{y^5}{x^5}$

5. Simplify using only positive exponents: $\dfrac{(2x^2y^{-5})^2}{x^2}$.

ans: $\dfrac{4x^2}{y^{10}}$

6. Simplify using only positive exponents: $\dfrac{x^{-4}x^{-3}}{y^{-7}y^6}$.

ans: $\dfrac{y}{x^7}$

7. Simplify using only positive exponents: $\dfrac{3ab^{-2}}{a^{-2}b^3}$.

ans: $\dfrac{3a^3}{b^5}$

8. Simplify using only positive exponents: $\left(\dfrac{x^{-3}}{y^2w^{-3}}\right)^{-1}$

ans: $\dfrac{x^3y^2}{w^3}$

9. Simplify: $\dfrac{8x^3y^2 - 4x^2y - 2xy}{2xy}$

ans: $4x^2y - 2x - 1$

10. Factor $x^2 - 7x + 6$.

ans: $(x-1)(x-6)$

11. Factor $-x^2 + 2x + 3$.

ans: $-(x-3)(x+1)$

12. Factor $121z - z^3$.

ans: $z(11-z)(11+z)$

13. Factor $100(h+1)^3 - (h+1)^5$.

ans: $(h+1)^3(11+h)(9-h)$

14. Factor $t^4 - 2t^2 + 1$.

ans: $(t^2-1)^2$

15. Simplify: $\dfrac{z^2-2z}{z^2+2z}\dfrac{z^2}{z-2}$

ans: $\dfrac{z^2}{z+2}$

16. Expand and simplify $\left(\dfrac{2}{3}\right)\left[\left(\dfrac{3}{2}\right)u(u-v)-3v(u+v)\right]$.

ans: $u^2-3uv-2v^2$

17. Simplify and write as a single fraction $\dfrac{1+1/x}{2-1/y}$.

ans: $\frac{y(x+1)}{x(2y-1)}$

18. Simplify and write as a single fraction with positive exponents $\dfrac{(1/x+3)^{-1}}{x}$.

ans: $\frac{1}{3x+1}$

19. Solve for x: $\dfrac{-5}{13}x-35=0$

ans: $x=-91$

20. Solve for y: $\dfrac{1}{4}y-\dfrac{3}{5}=\dfrac{1}{3}-y$

ans: $y=\frac{56}{75}$

21. Solve for x: $2z^2x-z^3=1$

ans: $x=\frac{1+z^3}{2z^2}$

22. Solve for t: $t^2-42=-11t$

ans: $t=3$, $t=-14$

23. Solve for r: $\dfrac{4}{3}r+1=\dfrac{1}{3}-\dfrac{1}{3}r-r^2$

ans: $r=-\frac{2}{3}$, $r=-1$

24. Solve for w: $5w^2-8w=-13$

ans: no real roots

25. Solve for x: $5x^2+16x+2=0$

ans: $x=\frac{-8\pm3\sqrt{6}}{5}$

26. Solve for x: $\dfrac{x^2+1}{x}=1$

ans: no real roots