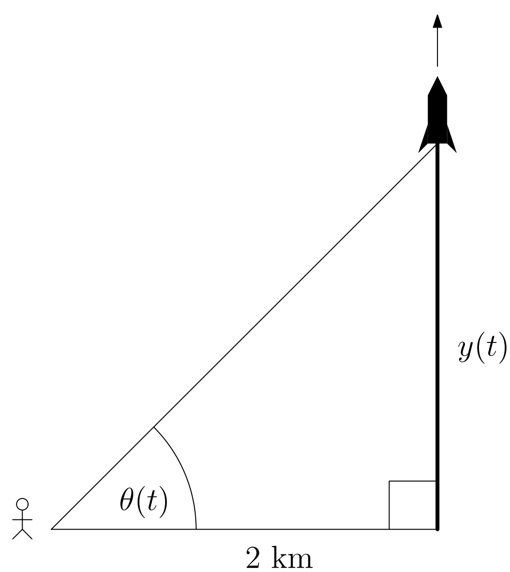
Related Rates Problems

In class we looked at an example of a type of problem belonging to the class of ***Related Rates Problems***: problems in which the rate of change (that is, the derivative) of an unknown function can be related to the rate of change of known functions. (Our example involved trigonometric function, but problems of related rates need not be restricted to only trig functions; functions of any type may be involved, but the principle remains the same.)

In this handout the example given in class is restated, followed by some exercises.

Example: An observer watches a rocket launch from a distance of 2 kilometres. The angle of elevation θ is increasing at $3°$ per second at the instant when $\theta = 45°$. How fast is the rocket climbing at that instant?

Solution: The figure below illustrates the situation:



Notice in the figure how we have labeled the angle of elevation $\theta(t)$ to remind ourselves that θ is a function of t. Also, we have labeled the height of the rocket as time t as $y(t)$, again emphasizing the dependence of y on time. Note also that no claim is made in the problem that the rocket is climbing at a constant rate; we only have information about the rate of change of θ at the instant when $\theta = 45°$.

Step 1 We should first notice that angle measure was given in degrees, but in calculus we always work in radians, so let's convert:

$$3^\circ = 3^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{60} \text{ radians}$$

$$45^\circ = 45^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{4} \text{ radians}$$

Step 2 We can now tackle this word problem. Read and understand the problem: done.

Step 3 Draw a diagram and introduce variable or function names: done.

Step 4 Summarize given data as concise calculus statements. We notice that the phrase "the angle of elevation ... is increasing..." is really a statement about a derivative:

$$\frac{d}{dt}\left(\theta(t)\right) = \frac{\pi}{60} \text{ radians per second when } \theta = \frac{\pi}{4} \text{ radians}$$

Step 5 Restate the question as concisely as possible (i.e. tell yourself what it is you are looking for):

$$\text{find } \frac{d}{dt}\left(y(t)\right) \text{ when } \theta = \frac{\pi}{4} \text{ radians}$$

Step 6 Find a relationship between the known and unknown functions. Here you want to know something about $y(t)$ while you know information about $\theta(t)$. From our work on trigonometric functions, at any point in time,

$$\tan\theta(t) = \frac{y(t)}{2}$$
$$\text{or } y(t) = 2\tan\theta(t)$$

Step 7 Finally, answer the question we asked ourselves in Step 5. Taking the derivative with respect to t of both sides of the last equation in the previous step (and remembering the chain rule!) gives

$$\begin{aligned}
\frac{d}{dt}\left[y(t)\right] &= 2\frac{d}{dt}\left[\tan\theta(t)\right] \\
&= 2\frac{1}{\cos^2\theta(t)}\frac{d}{dt}\left(\theta(t)\right) \\
\text{i.e. } \frac{dy}{dt} &= 2\frac{1}{\cos^2\theta}\frac{d\theta}{dt} ,
\end{aligned}$$

where in this last line we have dropped the t variable since we are through taking derivatives (y and θ are still functions of t however). Now at the instant when $\theta = \pi/4$, $d\theta/dt = \pi/60$, so

$$\begin{aligned}
\frac{dy}{dt} &= 2\frac{1}{\cos^2(\pi/4)}\frac{\pi}{60} \\
&= 2\frac{1}{(1/\sqrt{2})^2}\frac{\pi}{60} \\
&= \frac{\pi}{15}\,\frac{\text{km}}{\text{s}} \\
&\doteq 753\,\frac{\text{km}}{\text{hr}}
\end{aligned}$$

Step 8 Conclusion: the rocket is climbing at $\pi/15$ kilometres per second when θ is $45°$.

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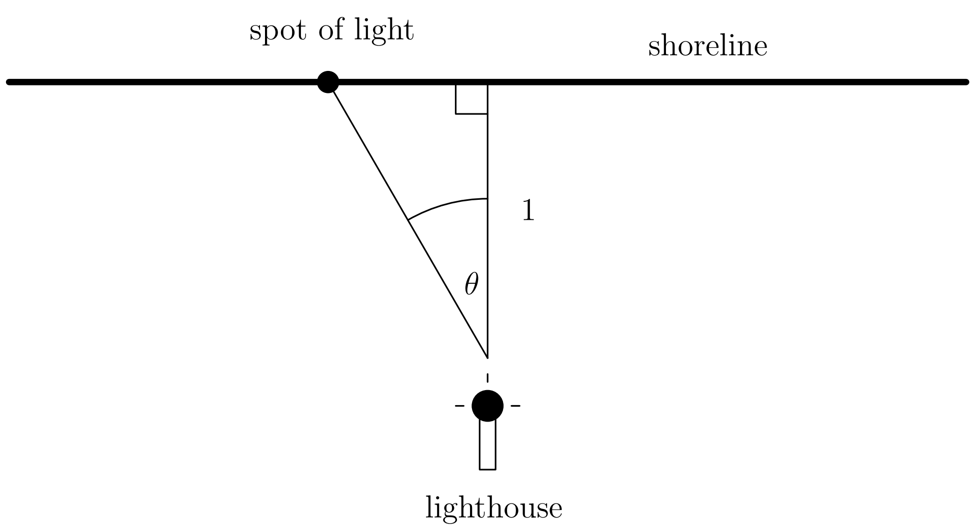
Notice in this problem that the numerical value for θ was substituted in only after we finished taking derivatives; this is important, and should be followed as a rule.

Exercises

1. Referring to the rocket example above, at what rate is the distance between the observer and the rocket increasing when $\theta = 45°$?

   ans: $\pi/(15\sqrt{2})$ km/s

2. The beacon of a lighthouse 1 km from a straight shore revolves 5 times per minute and shines a spot of light on the shore (see figure below).



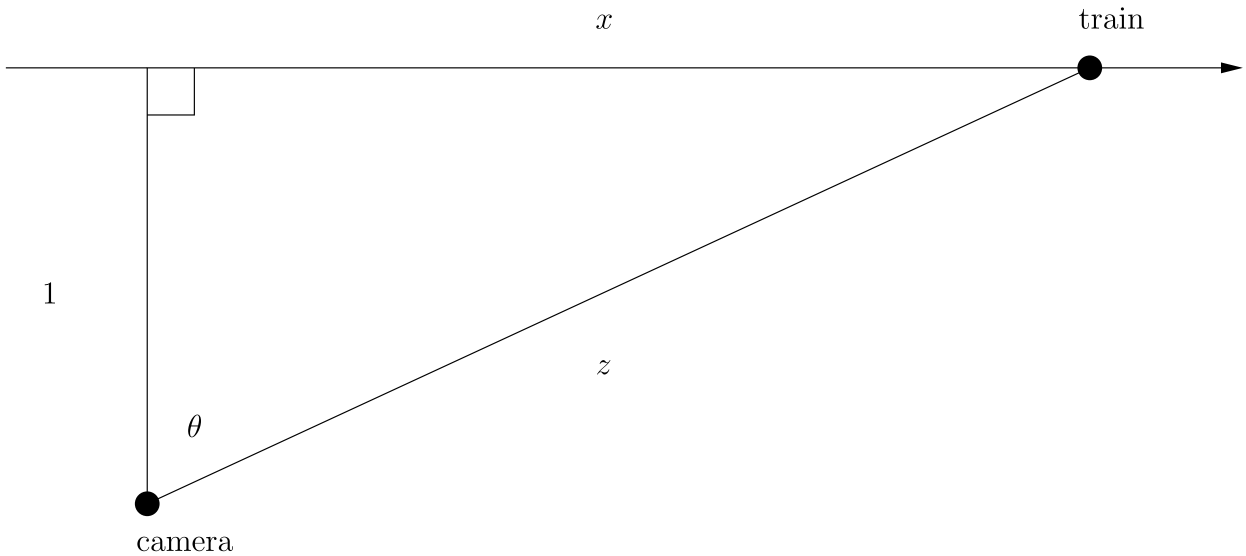
   (a) How fast is the spot of light moving when $\theta = 30°$?

   (b) What happens to the velocity of the spot of light as θ approaches $90°$?

   ans: (a) $40\pi/3$ km/min; (b) velocity grows to ∞

3. (More challenging) A train is traveling at $4/5$ km/min along a straight track, moving in the direction shown in the figure below. A movie camera positioned 1 km from the track is focused on the train.

   (a) How fast is the distance z from the camera to the train changing when the train is 2 km from the camera? (You can do this without trig: find an expression relating x and z, and remember these are functions of time.)



(b) When the train is 2 km from the camera $\theta = \pi/3$. How fast is the camera rotating (in radians/min) at the moment when the train is 2 km from the camera?

ans: (a) $2\sqrt{3}/5$ km/min; (b) $1/5$ radians/min