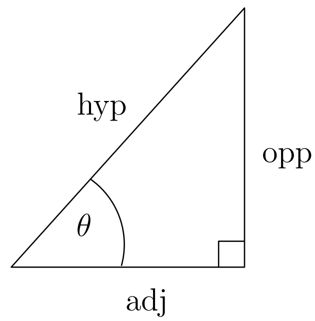
Introduction

With reference to the triangle



recall the basic trigonometric ratios sine, cosine and tangent for angles of less than 90°:

$$\boxed{\sin\theta = \frac{\text{opp}}{\text{hyp}}, \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}\ .}$$

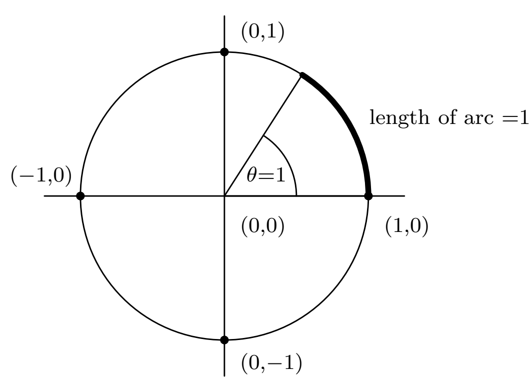
Note also that

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}\ .}$$

(The Greek letter θ (theta) is typically used to denote angle measure.)

Radian Measure and the Unit Circle

We are used to measuring angles in degrees, however there are other units of measure. In calculus we measure angles in ***radians***. To define what a radian is, we refer to the ***unit circle***, the circle of radius 1 centered at $(0,0)$, in which the angle θ shown is measured counterclockwise from the positive x-axis:



Notice: an angle of one radian sweeps out an arc of length one. The central angle of a circle measures 2π radians. That is,

$$\boxed{2\pi \text{ radians} = 360^\circ, \text{ or } \pi \text{ radians} = 180^\circ\ .}$$

Therefore to convert from degrees to radians, multiply by $\pi/180$. To convert from radians to degrees, multiply by $180/\pi$.

Example: Convert 125° to radians.
Solution:

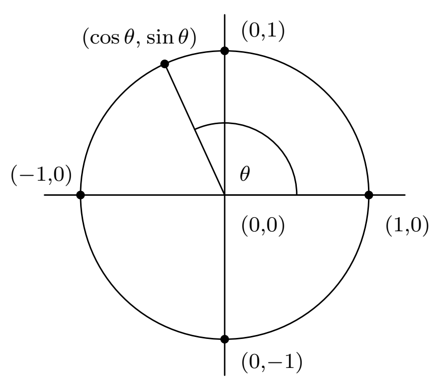
$$125^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{25\pi}{36} \text{ radians.}$$

(Notice how the old units cancel, leaving the units we want.) ■

IMPORTANT: In calculus, assume all angle measures are in radians unless told otherwise.

Extending the Definitions of the Trigonometric Functions

The definitions of $\sin, \cos$ and $\tan$ can be extended to angles of any size using the unit circle:



Notice: the ray which has swept out the angle θ has terminal point on the unit circle with x-coordinate $\cos\theta$ and y-coordinate $\sin\theta$. Observe the following:

- If $0 \le \theta < \pi/2$, this definition gives the basic trigonometric ratios we started with.
- If θ starts from 0 and increases to 2π, the point $(\cos\theta, \sin\theta)$ starts at $(1, 0)$ and moves around the unit circle, returning to $(1, 0)$. As θ increases from 2π to 4π, $(\cos\theta, \sin\theta)$ travels around the circle a second time, with $\cos\theta$ and $\sin\theta$ cycling through the same set of values. That is, $\cos\theta$ and $\sin\theta$ are ***periodic*** with period 2π:

$$\boxed{\cos(\theta + 2\pi) = \cos\theta, \qquad \sin(\theta + 2\pi) = \sin\theta}$$

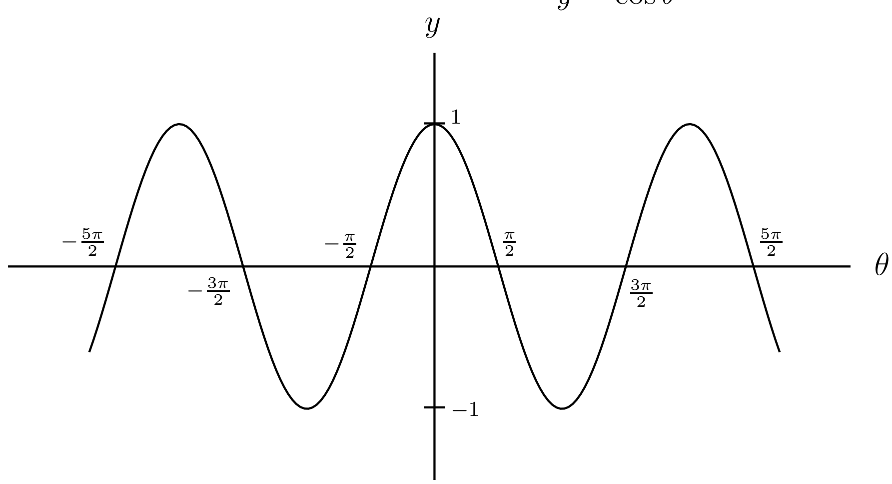
- From the unit circle we can easily read off the sine and cosine of some special angles. For example, $\theta = 3\pi/2$ corresponds to the point $(\cos\theta, \sin\theta) = (0, -1)$, so $\cos(3\pi/2) = 0$ and $\sin(3\pi/2) = -1$.

Using this extended definition of sine and cosine, the definition of the tangent function is then

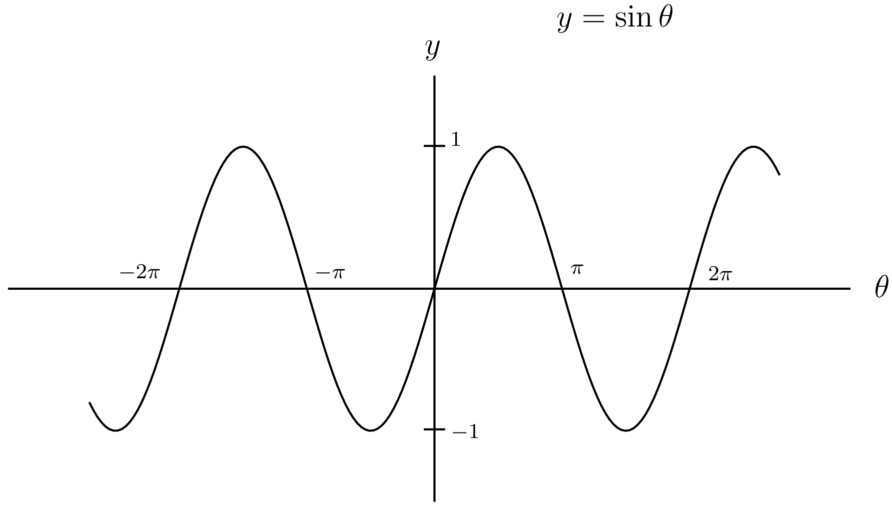
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \ .$$

The graphs of $y = \cos\theta$ and $y = \sin\theta$ are

$y = \cos\theta$



$y = \sin\theta$



Trigonometric Functions and Calculus

The trigonometric functions play an important role in calculus, particularly in the theory of differential equations. For now lets state the derivatives rules for the sine and cosine functions (here we'll use x instead of θ for the independent variable). These rules are valid as long as angles are measured in radians, otherwise the rules are different (and more complicated).

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

More generally, from the chain rule:

$$\boxed{\frac{d}{dx}(\sin f(x)) = \cos f(x) \cdot f'(x)}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin f(x) \cdot f'(x)}$$

Since sine and cosine are periodic, it is not surprising that their derivatives (i.e. slope of the tangent line) should be as well.

The derivative of the tan function can be found easily using the quotient rule:

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{(\cos x)^2} \\
&= \frac{\cos^2 x + \sin^2 x}{(\cos x)^2} \\
&= \frac{1}{\cos^2 x}
\end{aligned}$$

Here we've used the king of trigonometric identities: $\sin^2 x + \cos^2 x = 1$ for any real x. Also, the notation $\cos^2 x$ simply means $(\cos x)^2$. In summary,

$$\boxed{\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}}$$

and more generally, from the chain rule:

$$\boxed{\frac{d}{dx}(\tan f(x)) = \frac{1}{\cos^2 f(x)} \cdot f'(x)}$$

Let's do a few examples:

Example: Compute the derivative of $y = \sin(x^3 + x^{1/2})$
Solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\sin(x^3 + x^{1/2})\right)\\ &= \cos(x^3 + x^{1/2})\frac{d}{dx}\left(x^3 + x^{1/2}\right)\\ &= \cos(x^3 + x^{1/2})\left(3x^2 + \frac{1}{2}x^{-1/2}\right)\end{aligned}$$

■

Example: Compute the derivative of $y = e^{x^2}\cos(\pi x)$.
Solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(e^{x^2}\right)\cos(\pi x) + e^{x^2}\frac{d}{dx}(\cos(\pi x))\\ &= e^{x^2}2x\cos(\pi x) - e^{x^2}\sin(\pi x)\pi\end{aligned}$$

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Example: Find the tangent line to the curve $w = \tan(\pi e^t)$ at the point where $t = 0$.
Solution: At $t = 0$ $w = \tan(\pi e^0) = \tan\pi = 0$, so the tangent line passes through the point $(0, 0)$. Also,

$$w' = \frac{-1}{\cos^2(\pi e^t)}\pi e^t$$

which at $t = 0$ gives

$$w' = \frac{-1}{\cos^2(\pi e^0)}\pi e^0 = -\pi\ .$$

Therefore,

$$w - 0 = -\pi(t - 0)$$

or more simply, $w = -\pi t$. ■

Exercises

1. Find the derivative of $y = \sin(2x)\sin(3x)$.

   ans: $2\cos(2x)\sin(3x) + 3\sin(2x)\cos(3x)$

2. Compute the derivative: $y = \sqrt{\cos x} + \cos\sqrt{x}$.

   ans: $(-1/2)(\cos x)^{-1/2}\sin x - \sin\sqrt{x}(1/2)x^{-1/2}$

3. Compute the derivative: $y = \pi + \sin^2(3x) + \sin(3x)^2$.

   ans: $6\sin(3x)\cos(3x) + 18x\cos(3x)^2$

4. There are three other trigonometric functions we have not mentioned. These are defined as $\sec x = 1/\cos x$, $\csc x = 1/\sin x$, and $\cot x = 1/\tan x$. Use your derivative rules and the rules for sin, cos and tan to compute the derivatives of these new functions.

   ans: $\frac{d}{dx}(\sec x) = \sec x \tan x$; $\frac{d}{dx}(\csc x) = -\csc x \cot x$; $\frac{d}{dx}(\cot x) = -\csc^2 x$.

5. Let $f(\theta) = \ln(\cos\theta)$. Find the equation of the tangent line to the graph of f at $\theta = 2\pi$.

   ans: $y = 0$

6. Let $w = \tan(\pi \ln t)$. Compute $\left.\dfrac{dw}{dt}\right|_{t=e^2}$.

   ans: π/e^2

7. Let $y = x^2 + \sin x^2$. Find all x on the interval $[0, 2]$ where $y' = 0$.

   ans: $x = 0, \sqrt{\pi}$

8. Find the derivative of $(\sin\theta)^{\cos\theta}$.

   ans: $(\sin\theta)^{\cos\theta}(\cos^2\theta/\sin\theta - \sin\theta \ln(\sin\theta))$

9. Find the derivative of $e^{\theta}\sin 2\theta$.

   ans: $e^{\theta}\sin 2\theta + e^{\theta} 2\cos 2\theta$

10. Suppose x is measured in degrees. What then is the derivative of $\sin x$?

    ans: $(\pi/180)\cos x$