Linear Approximation

Introduction

By now we have seen many examples in which we determined the tangent line to the graph of a function $f(x)$ at a point $x = a$. A ***linear approximation*** (or ***tangent line approximation***) is the simple idea of using the equation of the tangent line to approximate values of $f(x)$ for x near $x = a$.

A picture really tells the whole story here. Take a look at the figure below in which the graph of a function $f(x)$ is plotted along with its tangent line at $x = a$. Notice how, near the point of contact $(a, f(a))$, the tangent line nearly coincides with the graph of $f(x)$, while the distance between the tangent line and graph grows as x moves away from a.

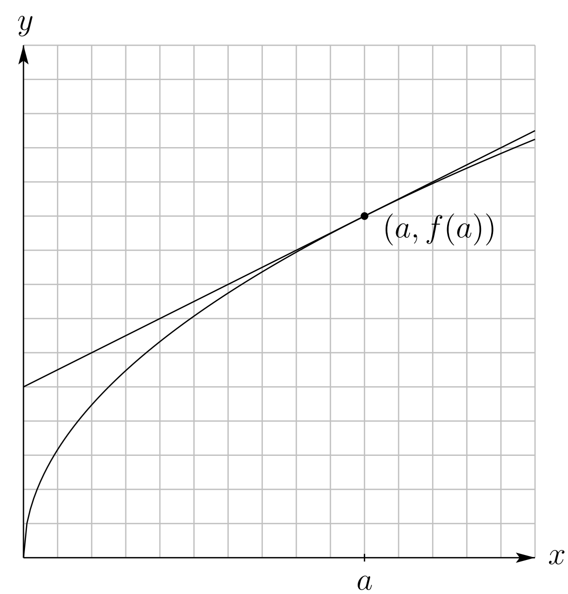

Figure 1: Graph of $f(x)$ with tangent line at $x = a$

In other words, for a given value of x close to a, the difference between the corresponding y value on the graph of $f(x)$ and the y value on the tangent line is very small.

The Linear Approximation Formula

Translating our observations about graphs into practical formulas is easy. The tangent line in Figure 1 has slope $f'(a)$ and passes through the point $(a, f(a))$, and so using the point-slope formula $y - y_0 = m(x - x_0)$, the equation of the tangent line can be expressed

$$y - f(a) = f'(a)(x - a),$$

or equivalently, isolating y,

$$y = f(a) + f'(a)(x - a)\ .$$

(Observe how this last equation gives us a new simple and efficient formula for the equation of the tangent line.) Again, the idea in linear approximation is to approximate the y values on the graph $y = f(x)$ with the y values of the tangent line $y = f(a) + f'(a)(x - a)$, so long as x is not too far away from a. That is,

$$\boxed{\text{for } x \text{ near } a,\ f(x) \approx f(a) + f'(a)(x - a)\ .} \qquad (1)$$

Equation (1) is called the linear approximation (or tangent line approximation) of $f(x)$ at $x = a$. (Instead of “at”, some books use “about”, or “near”, but it means the same thing.)

Notice how we use “$\approx$” instead of “$=$” to indicate that $f(x)$ is being approximated. Also notice that if we set $x = a$ in Equation (1) we get true equality, which makes sense since the graphs of $f(x)$ and the tangent line coincide at $x = a$.

A Simple Example

Let’s look at a simple example: consider the function $f(x) = \sqrt{x}$. The tangent line to $f(x)$ at $x = 1$ is $y = x/2 + 1/2$ (so here $a = 1$ is the x value at which we are finding the tangent line.) This is actually the function and tangent line plotted in Figure 1. So here, for x near $x = 1$,

$$\sqrt{x} \approx \frac{x}{2} + \frac{1}{2}\ .$$

To see how well the approximation works, let’s approximate $\sqrt{1.1}$:

$$\begin{aligned}\sqrt{1.1} &\approx \frac{1.1}{2} + \frac{1}{2}\\ &= 1.05\end{aligned}$$

Using a calculator, we find $\sqrt{1.1} \doteq 1.0488$ to four decimal places. So our approximation has an error of about 0.1%; not bad considering the simplicity of the calculation in the linear approximation!

On the other hand, if we try to use the same linear approximation for an x value far from $x = 1$, the results are not so good. For example, let's approximate $\sqrt{0.25}$:

$$\begin{aligned}\sqrt{0.25} &\approx \frac{0.25}{2} + \frac{1}{2} \\ &= 0.625\end{aligned}$$

The exact value is $\sqrt{0.25} = 0.5$, so our approximation has an error of 25%, a pretty poor approximation.

More Examples

Example 1: Find the linear approximation of $f(x) = x\sin(\pi x^2)$ about $x = 2$. Use the approximation to estimate $f(1.99)$.

Solution: Here $a = 2$ so we need $f(2)$ and $f'(2)$:

$$f(2) = 2\sin(4\pi) = 0,$$

while

$$f'(x) = \sin(\pi x^2) + x\cos(\pi x^2)\, 2\pi x\ ,$$

so that

$$f'(2) = \sin(4\pi) + 8\pi\cos(4\pi) = 8\pi\ .$$

Therefore the linear approximation is

$$f(x) \approx f(2) + f'(2)(x - 2)\ ,$$

i.e.

$$\text{for } x \text{ near } 2,\ x\sin(\pi x^2) \approx 8\pi(x - 2)\ .$$

Using this to estimate $f(1.99)$, we find

$$f(1.99) \approx 8\pi(1.99 - 2) = -0.08\pi \doteq -0.251$$

to three decimals. (Checking with a calculator we find $f(1.99) \doteq -0.248$ to three decimals.)

■

Example 2: Use a tangent line approximation to estimate $\sqrt[3]{28}$ to 4 decimal places.

Solution: In this example we must come up with the appropriate function and point at which to find the equation of the tangent line. Since we wish to estimate $\sqrt[3]{28}$, $f(x) = x^{1/3}$. For the a-value in Equation (1) we ask: at what value of x near 28 do we know $f(x)$ exactly? Answer: $x = 27$, which is a perfect cube.

Thus, using $f(x) = x^{1/3}$ we find $f'(x) = (1/3)x^{-2/3}$, so that $f(27) = 3$ and $f'(27) = 1/27$. The linear approximation formula is then

$$f(x) \approx f(27) + f'(27)(x - 27) \ ,$$

i.e., for x near 27,

$$x^{1/3} \approx 3 + \frac{1}{27}(x - 27) \ .$$

Using this to approximate $\sqrt[3]{28}$ we find

$$\begin{aligned} \sqrt[3]{28} &\approx 3 + \frac{1}{27}(28 - 27) \\ &= \frac{82}{27} \\ &\doteq 3.0370 \end{aligned}$$

A calculator check gives $\sqrt[3]{28} \doteq 3.0366$ to 4 decimals.

■

Example 3: Consider the implicit function defined by

$$3(x^2 + y^2)^2 = 100xy \ .$$

Use a tangent line approximation at the point $(3, 1)$ to estimate the value of y when $x = 3.1$.

Solution: Even though y is defined implicitly as a function of x here, the tangent line to the graph of $3(x^2 + y^2)^2 = 100xy$ at $(3, 1)$ can easily be found and used to estimate y for x near 3.

First, find y'. Differentiating both sides of $3(x^2 + y^2)^2 = 100xy$ with respect to x gives

$$6(x^2 + y^2)(2x + 2yy') = 100y + 100xy' \ .$$

Now substitute $(x, y) = (3, 1)$:

$$6(9 + 1)(6 + 2y') = 100 + 300y'$$

which yields $y' = 13/9$. Thus the equation of the tangent line is

$$\begin{aligned} y - 1 &= \frac{13}{9}(x - 3), \text{ or} \\ y &= \frac{13}{9}x - \frac{30}{9} \ . \end{aligned}$$

Thus, for points (x, y) on the graph of $3(x^2 + y^2)^2 = 100xy$ with x near 3,

$$y \approx \frac{13}{9}x - \frac{30}{9}\ .$$

Setting $x = 3.1$ in this last equation gives $y \approx 103/90 \doteq 1.14$ to two decimals.

■

Exercises

1. Physicists often use the approximation $\sin x \approx x$ for small x. Convince yourself that this is valid by finding the linear approximation of $f(x) = \sin x$ at $x = 0$.

   Solution For x near 0, $f(x) \approx f(0) + f'(0)(x - 0)$. Using $f(x) = \sin x$, $f(0) = \sin(0) = 0$ and $f'(0) = \cos(0) = 1$ we find $\sin x \approx x$.

2. Find the linear approximation of $f(x) = x^3 - x$ about $x = 1$ and use it to estimate $f(0.9)$.

   Solution For x near 1, $f(x) \approx f(1) + f'(1)(x - 1)$. Using $f(x) = x^3 - x$, $f(1) = 0$ and $f'(1) = 2$ we find $f(x) \approx 2(x - 1)$, so $f(0.9) \approx 2(0.9 - 1) = -0.2$.

3. Use a linear approximation to estimate $\cos 62°$ to three decimal places. Check your estimate using your calculator. For this problem recall the trig value of the special angles:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$

   Solution Here $62°$ is close to $60°$ which is $\pi/3$ radians, and we know $\cos(\pi/3) = 1/2$. Letting $f(x) = \cos x$, for x near $\pi/3$, $f(x) \approx f(\pi/3) + f'(\pi/3)(x - \pi/3)$. Since $62° = 62\pi/180$ radians and $f'(x) = -\sin x$, this gives

$$\begin{aligned} \cos 62° &\approx 1/2 - \sin(\pi/3)(62\pi/180 - \pi/3) \\ &= 1/2 - (\sqrt{3}/2)(\pi/90) \\ &\doteq 0.470 \end{aligned}$$

4. Use a tangent line approximation to estimate $\sqrt[4]{15}$ to three decimal places.

   Solution 15 is near 16 where we know $\sqrt[4]{16} = 2$ exactly. Letting $f(x) = \sqrt[4]{x}$, we have for x near 16, $f(x) \approx f(16) + f'(16)(x - 16)$. That is, $\sqrt[4]{x} \approx 2 + (1/32)(x - 16)$. Thus

$$\begin{aligned} \sqrt[4]{x} &\approx 2 + (1/32)(15 - 16) \\ &= 63/32 \\ &\doteq 1.969\ . \end{aligned}$$

5. Define y implicitly as a function of x via $x^{2/3} + y^{2/3} = 5$. Use a tangent line approximation at $(8, 1)$ to estimate the value of y when $x = 7$.

   Solution First find the equation of the tangent line to the curve at $(8, 1)$ and then substitute $x = 7$. Differentiating implicity with respect to x we find

   $$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

   and substituting $(x, y) = (8, 1)$ yields $y' = -1/2$. Thus the equation of the tangent line is

   $$y = 1 - \frac{1}{2}(x - 8)$$

   and substituting $x = 7$ we find $y = 3/2$. That is, $(7, 3/2)$ is the point on the tangent line. Thus the point on the curve with x coordinate $x = 7$ has corresponding y coordinate $y \approx 3/2$.

6. Suppose $f(x)$ is a differentiable function whose graph passes through the points $(-1, 4)$ and $(1, 7)$. The estimate $f(-0.8) \approx 5$ is obtained using a linear approximation about $x = -1$. Using this information, find $\left.\frac{d}{dx}(f(x))^2\right|_{x=-1}$.

   Solution This problem was made more difficult by adding extra information which is not needed for the solution: the point $(1, 7)$ plays no part. First, note that since $(-1, 4)$ is on the graph of $f(x)$, $f(-1) = 4$. For x near -1, $f(x) \approx f(-1) + f'(-1)(x + 1)$. Using this linear approximation, the estimate $f(-0.8) \approx 5$ was made; that is

   $$5 = 4 + f'(-1)(-0.8 + 1)$$

   So that $f'(-1) = 5$. Now do the derivative, remembering the chain rule:

   $$\begin{aligned} \left.\frac{d}{dx}(f(x))^2\right|_{x=-1} &= \left.2f(x)f'(x)\right|_{x=-1} \\ &= 2(4)(5) \\ &= 40 \;. \end{aligned}$$

7. The profit $P(q)$ from producing q units of goods is given by

   $$P(q) = 396q - 2.2q^2 + k$$

   for some constant k. Using a linear approximation about $q = 80$ we find $P(81) \approx 17244$. What is k?

   Solution For q near 80, $P(q) \approx P(80) + P'(80)(q - 80)$. Using this approximation, $P(81) \approx 17244$, so that

   $$\begin{aligned} 17244 &= P(80) + P'(80)(q - 80) \\ 17244 &= [396(80) - 2.2(80)^2 + k] + [396 - 4.4(80)](1) \end{aligned}$$

   where in this last equation the first expression in square brackets is $P(80)$ and the second expression in square brackets is $P'(80)$. Solving this last equation for k gives $k = -400$ (note the original answers had $k = 400$ which is incorrect).