Division involving 0 may be summarized as follows. Let n be any non-zero whole number. Then,

1. $n \div 0$ is undefined. 2. $0 \div n = 0$. 3. $0 \div 0$ is undefined.

In general, division by 0 is undefined.

Recall that $n \cdot 1 = n$ for any whole number n. Thus, by the definition of division, $n \div 1 = n$. For example, $3 \div 1 = 3$, $1 \div 1 = 1$, and $0 \div 1 = 0$.

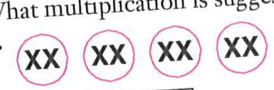
Order of Operations

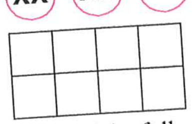
Difficulties involving the order of arithmetic operations sometimes arise. For example, many students treat $2 + 3 \cdot 6$ as $(2 + 3)6$, whereas others treat it as $2 + (3 \cdot 6)$. In the first case, the value is 30; in the second case, the value is 20. To avoid confusion, mathematicians agree that when no parentheses are present, multiplications and divisions are performed *before* additions and subtractions. The multiplications and divisions are performed in the order they occur from left to right, and then the additions and subtractions are performed in the order they occur from left to right. Thus, $2 + 3 \cdot 6 = 2 + 18 = 20$. If parentheses are present, the computations in parentheses are done first. This order of operations is not built into calculators that display an incorrect answer of 30.

The computation $8 - 9 \div 3 \cdot 2 + 3$ is performed as follows.

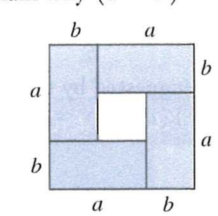
$$\begin{aligned} 8 - 9 \div 3 \cdot 2 + 3 &= 8 - 3 \cdot 2 + 3 \\ &= 8 - 6 + 3 \\ &= 2 + 3 \\ &= 5 \end{aligned}$$

Assessment 3-3A

1. Show that $3 \cdot 4 = 12$ using each of the following models.
   a. Repeated-addition number line
   b. Rectangular array
   c. Area
   d. Cartesian product
2. If $A = \{a, b\}$ and $B = \{x, y, z\}$, find each of the following.
   a. $A \times B$
   b. $n(A \times B)$
   c. Write a multiplication equation using numerals related to the answers in parts (a) and (b).
3. What multiplication is suggested by the following models?
   a. 

   b. 
4. For each of the following find, if possible, a whole number that makes the equation true.
   a. $3 \cdot \square = 15$ b. $18 = 6 + 3 \cdot \square$
   c. $\square \cdot (5 + 6) = \square \cdot 5 + \square \cdot 6$
5. Determine whether the following sets are closed under multiplication.
   a. $\{0, 1\}$ b. $\{2, 4, 6, 8, 10, \ldots\}$
   c. $\{1, 4, 7, 10, 13, \ldots\}$
6. a. If 5 is removed from the set of whole numbers, is the set closed with respect to addition? Explain.
   b. If 5 is removed from the set of whole numbers, is the set closed with respect to multiplication? Explain.
7. Rename each of the following using the distributive property of multiplication over addition so there are no parentheses in the final answer:
   a. $(a + b)(c + d)$
   b. $\square(\Delta + \bigcirc)$
   c. $a(b + c) - ac$
8. Place parentheses, if needed, to make each of the following equations true.
   a. $5 + 6 \cdot 3 = 33$
   b. $8 + 7 - 3 = 12$
   c. $6 + 8 - 2 \div 2 = 13$
   d. $9 + 6 \div 3 = 5$
9. Using the distributive property of multiplication over addition, we can factor as in $x^2 + xy = x(x + y)$. Use the distributive property and other multiplication properties to factor each of the following expressions.
   a. $xy + y^2$ b. $xy + x$
   c. $a^2b + ab^2$
10. For each of the following equations, find whole numbers to make the statement true, if possible.
   a. $18 \div 3 = \square$ b. $\square \div 76 = 0$
   c. $28 \div \square = 7$

11. A sporting goods store has designs for 6 shirts, 4 pairs of pants, and 3 vests. How many different shirt-pants-vest outfits are possible?
12. Which property is illustrated in each of the following equations.
   a. $6(5 \cdot 4) = (6 \cdot 5)4$
   b. $6(5 \cdot 4) = 6(4 \cdot 5)$
   c. $6(5 \cdot 4) = (5 \cdot 4)6$
   d. $1 \cdot (5 \cdot 4) = 5 \cdot 4$
   e. $(3 + 4) \cdot 0 = 0$
   f. $(3 + 4)(5 + 6) = (3 + 4)5 + (3 + 4)6$
13. Students are overheard making the following statements. What properties justify their statements?
   a. I know that $9 \cdot 7$ is either 63 or 69, and I know they can't both be right.
   b. I know that $9 \cdot 0$ is 0 because I know that any number times 0 is 0.
   c. Any number times 1 is the same as the number we started with, so $9 \cdot 1$ is 9.
14. The product $6 \cdot 14$ can be found by thinking of the problem as $6(10 + 4) = 6 \cdot 10 + 6 \cdot 4 = 60 + 24 = 84$.
   a. What properties are being used?
   b. Use this technique to mentally compute $32 \cdot 12$.
15. Use the distributive property of multiplication over subtraction to compute each of the following expressions.
   a. $9(10 - 2)$
   b. $20(8 - 3)$
16. Show that $(a + b)^2 = a^2 + 2ab + b^2$ using
   a. the distributive property of multiplication over addition and other properties.
   b. an area model.
17. If a and b are whole numbers with $a > b$, use the rectangles in the figure to explain why $(a + b)^2 - (a - b)^2 = 4ab$.

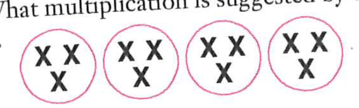


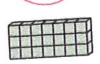
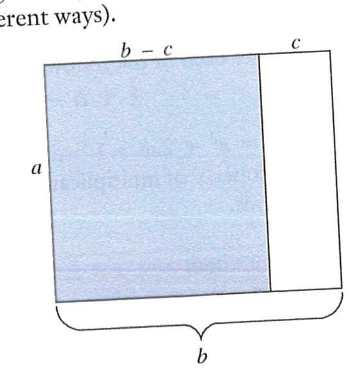
18. Use the property $(a + b)^2 = a^2 + 2ab + b^2$ to compute the following expressions.
   a. 51^2 b. 102^2
19. In each of the following equations, show that the left side of the equation is equal to the right side and give a reason for every step.
   a. $(ab)c = (ca)b$ b. $(a + b)c = c(b + a)$
20. Factor each of the following expressions.
   a. $xy - y^2$
   b. $47 \cdot 101 - 47$
   c. $ab^2 - ba^2$
21. Rewrite each of the following division problems as a multiplication problem.
   a. $40 \div 8 = 5$
   b. $326 \div 2 = x$
22. If $108/a = b$, then find $108/b$.
23. Write the complete fact family for $72/8 = 9$.
24. Think of a number. Multiply it by 5. Add 5. Divide by 5 and then subtract 1. How does the result compare with your original number? Will this work all the time? Justify your answer.
25. Show that, in general, each of the following is false if a, b, and c are whole numbers.
   a. $(a \div b) \div c = a \div (b \div c)$
   b. $a \div (b + c) = (a \div b) + (a \div c)$
26. Suppose all of the operations result in whole numbers. Explain why $(a + b) \div c = (a \div c) + (b \div c)$.
27. Find the solution for each of the following equations.
   a. $5x + 2 = 22$ b. $3x + 7 = x + 13$
   c. $3(x + 4) = 18$ d. $(x - 5) \div 10 = 9$
28. A new model of car is available in 4 exterior colors and 3 interior colors. Use a tree diagram and specific colors to show how many color schemes are possible for the car.
29. Is it possible to find a whole number less than 100 that when divided by 10 has remainder 4 and when divided by 47 has remainder 17?
30. Students were divided into 10 teams with 12 on each team. Later, the same students were divided into teams with 8 on each team. How many teams were there then?
31. In each of the following, tell what computation must be done last:
   a. $5(16 - 7) - 18$
   b. $54/(10 - 5 + 4)$
   c. $(14 - 3) + (24 \cdot 2)$
   d. $21{,}045/345 + 8$
32. Find infinitely many whole numbers that leave remainder 1 upon division by 4.
33. The operation $\odot$ is defined on the set $S = \{a, b, c\}$, as shown in the following table. For example, $a \odot b = b$ and $b \odot a = b$.

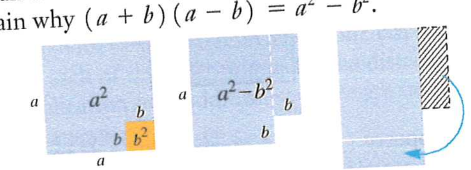
$\odot$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

   a. Is S closed with respect to $\odot$?
   b. Is $\odot$ commutative on S?
   c. Is there an identity for $\odot$ on S? If yes, what is it?
   d. Try several examples to investigate the associative property for $\odot$ on S.
34. If $a, b \in W$, is the operation # commutative if $a \# b = a + b + 4$? Explain why.
35. At a certain concert, the audience was allowed to enter in a certain way. The first time a bell rang only 1 person was allowed to enter and choose his or her seat. The second time the bell rang 3 people were allowed to enter. On each successive ring the group that enters has two more people than the previous group.
   a. How many people have entered after the 25th ring of the bell?
   b. How many have entered after the nth ring?
   c. After how many rings will there be at least 1000 people?

Assessment 3-3B

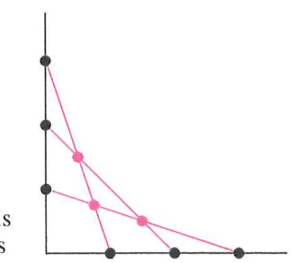
1. Show that $5 \cdot 3 = 15$ using each of the following models.
   a. Repeated-addition number line
   b. Rectangular array
   c. Area
   d. Cartesian product model
2. If $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$, find each of the following.
   a. $A \times B$ b. $n(A \times B)$
   c. Write a multiplication equation involving numerals related to the answers in parts (a) and (b).
3. What multiplication is suggested by the following models?
   a.


   b.

4. For each of the following, find, if possible, a whole number that makes the equation true.
   a. $8 \cdot \square = 24$ b. $28 = 4 + 6 \cdot \square$
   c. $\square \cdot (8 + 6) = \square \cdot 8 + \square \cdot 6$
5. Determine if the following sets are closed under multiplication.
   a. $\{1, 2\}$
   b. $\{2k + 1 \mid k \in W\}$
   c. $\{2^{k+1} \mid k \in W\}$
6. a. If 2 is removed from the set of whole numbers, is the set closed with respect to addition? Explain.
   b. If 1 is removed from the set of whole numbers, is the set closed with respect to multiplication? Explain.
7. Rename each of the following using the distributive property of multiplication over addition so there are no parentheses in the final answer. Use other properties to simplify when possible.
   a. $3(x + y + 5)$
   b. $(x + y)(x + y + z)$
   c. $x(y + 1) - x$
8. Place parentheses, if needed, to make each of the following equations true:
   a. $4 + 3 \cdot 2 = 14$
   b. $9 \div 3 + 1 = 4$
   c. $5 + 4 + 9 \div 3 = 6$
   d. $3 + 6 - 2 \div 1 = 7$
9. Using the distributive property of multiplication over addition, we can factor as in $x^2 + xy = x(x + y)$. Use the distributive property and other multiplication properties to factor each of the following.
   a. $47 \cdot 99 + 47$
   b. $(x + 1)y + (x + 1)$
   c. $x^2y + zx^3$
10. For each of the following equations, find whole numbers to make the statement true, if possible.
   a. $27 \div 9 = \square$ b. $\square \div 52 = 1$
   c. $13 \div \square = 13$
11. A new car comes in 5 exterior colors and 3 interior colors. How many different looking cars are possible?
12. Which property of whole numbers is illustrated in each of the following equations?
   a. $(5 \cdot 4)0 = 0$
   b. $7(3 \cdot 4) = 7(4 \cdot 3)$
   c. $7(3 \cdot 4) = (3 \cdot 4)7$
   d. $(3 + 4)1 = 3 + 4$
   e. $(3 + 4)5 = 3 \cdot 5 + 4 \cdot 5$
   f. $(1 + 2)(3 + 4) = (1 + 2)3 + (1 + 2)4$
13. Students are overheard making the following statements. What properties justify their statements?
   a. I know if I remember what $7 \cdot 9$ is, then I also know what $9 \cdot 7$ is.
   b. To find $9 \cdot 6$, I just remember that $9 \cdot 5$ is 45, so $9 \cdot 6$ is just 9 more than 45, or 54.
14. The product $5 \cdot 24$ can be found by thinking of the computation as $5(20 + 4) = 5 \cdot 20 + 5 \cdot 4 = 100 + 20 = 120$.
   a. What property is being used?
   b. Use this technique to mentally compute $8 \cdot 34$.
15. Use the distributive property of multiplication over subtraction to compute each of the following.
   a. $15(10 - 2)$ b. $30(9 - 2)$
16. Show that if $b > c$, then $a(b - c) = ab - ac$ using an area model suggested by the given figure (express the shaded area in two different ways).


17. Use an area model suggested by the following figure to explain why $(a + b)(a - b) = a^2 - b^2$.


18. Use the formula $(a + b)(a - b) = a^2 - b^2$ to compute the following.
   a. $19 \cdot 21$ b. $25 \cdot 15$
   c. $99 \cdot 101$ d. $101^2 - 99^2$
19. Show that the left-hand side of the equation is equal to the right-hand side and give a reason for every step.
   a. $(ab)c = b(ac)$ b. $a(b + c) = ac + ab$
20. Factor each of the following:
   a. $xy - y$
   b. $(x + 1)y - (x + 1)$
   c. $a^2b^3 - ab^2$
21. Rewrite each of the following division problems as a multiplication problem.
   a. $48 \div x = 16$
   b. $x \div 5 = 17$
22. If $\frac{64}{a} = b$, then find $\frac{64}{b}$.
23. Write the complete fact family for $30/6 = 5$.

24. Think of a number. Multiply it by 2. Add 2. Divide by 2. Subtract 1. **(a)** How does the result compare with your original number? **(b)** Will this work all the time? Explain your answer.
25. Show that, in general, each of the following is false if a and b are whole numbers.
   a. $a \div b = b \div a$ b. $a - b = b - a$
26. Suppose all operations result in whole numbers. Explain why $(a - b) \div c = (a \div c) - (b \div c)$.
27. Find the solution for each of the following equations.
   a. $5x + 8 = 28$ b. $5x + 6 = x + 14$
   c. $5(x + 3) = 35$ d. $(x - 6) \div 3 = 1$
28. String art is formed by connecting evenly spaced nails on the vertical and horizontal axes by segments of string. Connect the nail farthest from the origin on the vertical axis with the nail closest to the origin on the horizontal axis. Continue until all nails are connected, as shown in the figure that follows. How many intersection points (red dots) are created with 10 nails on each axis?


3 nails per axis
3 intersections

29. Jonah has a large collection of marbles. He notices that if he borrows 5 marbles from a friend, he can arrange the marbles in rows of 13 each. What is the remainder when he divides his original number of marbles by 13?
30. Students were divided into eight teams with nine on each team. Later, the same students were divided into teams with six on each team. How many teams were there then?
31. In each of the following, tell what computation must be done last.
   a. $5 \cdot 6 - 3 \cdot 4 + 2$
   b. $19 - 3 \cdot 4 + 9 \div 3$
   c. $15 - 6 \div 2 \cdot 4$
   d. $5 + (8 - 2)3$
32. Find infinitely many whole numbers that leave remainder 3 upon division by 5.
33. The operation $\odot$ is defined on the set $S = \{a, b, c\}$, as shown in the following table. For example, $a \odot b = b$ and $b \odot a = b$.

$\odot$	a	b	c
a	a	b	c
b	b	a	c
c	c	c	c

   a. Is S closed with respect to $\odot$?
   b. Is $\odot$ commutative on S?
   c. Is there an identity for $\odot$ on S? If yes, what is it?
   d. Try several examples to investigate the associative property for $\odot$ on S.
34. If $a, b \in W$, is the operation # commutative if $a \# b = 0$, if either a or b or both are odd and $a \# b = 1$, if a and b are both even?
35. When a marching band was placed in rows of 5, one member remained. When the members were placed in rows of 6, there was still one member remaining. However, when they were placed in rows of 7, nobody remained. What is the least number of members that could have been in the band?

Mathematical Connections 3-3

Reasoning

1. Why is an odd number times an odd number always odd?
2. Can 0 be the identity for multiplication? Explain why or why not.
3. Suppose you forgot the product of $9 \cdot 7$. Give several ways that you could find the product using different multiplication facts and properties.
4. Is $x \div x$ always equal to 1? Explain your answer.
5. Is $x \cdot x$ ever equal to x? Explain your answer.
6. Describe all pairs of whole numbers whose sum and product are the same.
7. In the *Common Core Standards* quote at the beginning of this section it states that students should learn the relationship between multiplication and division. Explain this relationship.
8. Explain how the distributive property of multiplication over addition would be helpful to mentally perform the following computation.

$$37 \cdot 52 + 37 \cdot 48$$

Open-Ended

9. Create word problems using each model for multiplication and division.
10. Describe a real-life situation that could be represented by the expression $3 + 2 \cdot 6$.

Cooperative Learning

11. Multiplication facts that most children have memorized can be stated in the following table that is partially filled.

×	1	2	3	4	5	6	7	8	9
1									
2									
3									
4				16					
5							35		
6									
7									
8									72
9									81