New global exponential stability criteria for nonlinear delay differential systems with applications to BAM neural networks

Leonid Berezansky a, Elena Braverman b,⇑, Lev Idels c

a Dept. of Math, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
b Dept. of Math & Stats, University of Calgary, 2500 University Dr. NW, Calgary, AB T2N1N4, Canada
c Dept. of Math, Vancouver Island University (VIU), 900 Fifth St., Nanaimo, BC V9S5J5, Canada

ABSTRACT

We consider a nonlinear non-autonomous system with time-varying delays

\[ \dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{m} F_{ij}(t, x_j(g_{ij}(t))), \quad t \geq 0, \quad i = 1, \ldots, m \]

which has a large number of applications in the theory of artificial neural networks. Via the M-matrix method, easily verifiable sufficient stability conditions for the nonlinear system and its linear version are obtained. Application of the main theorem requires just to check whether a matrix, which is explicitly constructed using the system’s parameters, is an M-matrix. Comparison with the tests obtained by Gopalsamy (2007) and Liu (2013) for BAM neural networks illustrates novelty of the stability theorems. Some open problems conclude the paper.

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1. Introduction

One of the main motivations to study the nonlinear delay differential system

\[ \dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{m} F_{ij}(t, x_j(g_{ij}(t))), \quad t \geq 0, \quad i = 1, \ldots, m \]

(1.1)

and its linear version

\[ \dot{x}_i(t) = \sum_{j=1}^{m} a_{ij}(t)x_j(t), \quad i = 1, \ldots, m \]

(1.2)

is their importance in the study of artificial neural network models [12,14]. Here \( m \) is the number of units in a neural network, \( x_i(t) \in \mathbb{R}^m \) corresponds to the state vector of the \( i \)th unit at the time \( t \), and \( a_i(t) \) represents passive decay rates, \( F_{ij} \) are the

⇑ Corresponding author.
E-mail addresses: maelena@math.ucalgary.ca, maelena@ucalgary.ca (E. Braverman).
output transfer functions, \( h_i(t) \) denote the leakage (self-inflicted) delays, and \( g_{ij}(t) \) are transmission delays for the \( i \)th unit along the axon of the \( j \)th unit; for more details see, for example, [12,14].

For linear system (1.2) several very interesting stability results were obtained in [7,11,26,27]. In [11] system (1.2) with constant coefficients \( a_{ij} \) was examined; in [27] the proofs were based on the assumption that \( a_{ij} \) and \( g_{ij} \) are continuous functions and \( |a_{ij}| \leq |g_{ij}| \). Most of the results for system (1.1) were obtained in the case \( h_i(t) \equiv t \) (see, for example, [21]). Also the requirement that all the functions involved in the system are continuous seem unduly restrictive, and we relax this assumption. In the present paper, we consider the so-called pure-delay case \( h_i(t) \neq t \), assuming that all parameters are measurable functions, and \( F_{ij}(u) \) are Carathéodory functions. Via M-matrix method we obtain novel stability results for nonlinear non-autonomous system (1.1) and linear non-autonomous system (1.2). It is to be emphasized that our technique does not require a long sequence of other theorems or conditions that must be proven or cited before the main result is justified.

Gopalsamy in [10] studied a model of networks known as Bidirectional Associative Memory (BAM) with leakage delays:

\[
\begin{align*}
\dot{x}_i(t) &= -a_i x_i(t - \tau_i^{(1)}) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t - \sigma_j^{(2)})) + I_i \\
\dot{y}_j(t) &= -b_j y_j(t - \tau_j^{(2)}) + \sum_{i=1}^{m} b_{ij} g_i(x_i(t - \sigma_i^{(1)})) + J_j
\end{align*}
\]

This model can be regarded as a network with two layers, \( x_i(t) \) is the state vector of the \( i \)th neuron in the first layer, whereas \( y_j(t) \) refers to the second layer; \( a_i \) and \( b_j \) denote the neuron charging time constants and passive decay rates, respectively; \( a_{ij}, b_{ij} \) are synaptic connection strength; \( \tau_i^{(k)}, \sigma_j^{(k)} (k = 1, 2) \) are the leakage and the transmission delays accordingly, and \( I_i, J_j \) are the exogenous inputs [21].

The paper [10] contains sufficient conditions for the existence of a unique equilibrium of (1.3) and its global stability. Some interesting results for system (1.3) were obtained via the construction of Lyapunov functionals in [6,18,22,23,28,29].

To extend and improve the results obtained in [10,18,20], we apply our main theorem to the non-autonomous system

\[
\begin{align*}
\dot{x}_i(t) &= r_i(t) \left[ -a_i x_i(h_i^{(1)}(t)) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t - \sigma_j^{(2)})) + I_i \right] \\
\dot{y}_j(t) &= p_j(t) \left[ -b_j y_j(h_j^{(2)}(t)) + \sum_{i=1}^{m} b_{ij} g_i(x_i(t - \sigma_i^{(1)})) + J_j \right]
\end{align*}
\]

Let us quickly sketch what we accomplish here. Section 2 incorporates the main result of the paper: if a certain matrix which is explicitly constructed from the functions and the coefficients of the system is an M-matrix, then the system is globally exponentially stable. It is demonstrated that the stability condition for a nonlinear system of two equations with constant delays improves the test obtained in [10]. In Section 3 we examine stability of BAM models and obtain stability results that for a nonlinear BAM systems generalize the main theorem in [18]. Finally, Section 4 contains discussion and outlines some open problems.

2. Main results

Consider for any \( t_0 \geq 0 \) the system of delay differential equations

\[
\begin{align*}
\dot{x}_i(t) &= -a_i(t) x_i(h_i(t)) + \sum_{j=1}^{m} F_{ij}(t,x_j(g_{ij}(t))) , \quad t \geq t_0 , \quad i = 1, \ldots, m ,
\end{align*}
\]

with the initial conditions

\[
\begin{align*}
x_i(t) &= \varphi_i(t) , \quad t_0 - \sigma \leq t < t_0 , \quad x_i(t_0) = x_i^0 , \quad \text{where} \quad \sigma > 0 \text{ is denoted below in (a4), under the following assumptions:}
\end{align*}
\]

(a1) \( a_i \) are Lebesgue measurable essentially bounded on \( [0, \infty) \) functions, \( 0 < a_i(t) \leq A_i \) almost everywhere (a.e.);

(a2) \( F_{ij}(t,\cdot) \) are continuous functions, \( F_{ij}(\cdot,u) \) are measurable locally essentially bounded functions, \( |F_{ij}(t,u)| \leq L_{ij}|u| \) for a.e. \( t \geq 0 \);

(a3) \( h_i, g_{ij} \) are Lebesgue measurable functions, \( 0 \leq t - h_i(t) \leq \tau_i , \quad 0 \leq t - g_{ij}(t) \leq \sigma_{ij} \);

(a4) \( \varphi_i \) are continuous functions on \( [t_0 - \sigma, t_0] \), where \( \sigma = \max(\tau_k, \sigma_{ij}|k,i,j = 1, \ldots, m) \).

Further on we assume that all the inequalities for Lebesgue measurable functions (coefficients and delays) are satisfied a.e.

**Definition 2.1.** An absolutely continuous in \( [t_0, \infty) \) vector-function \( X = (x_1, \ldots, x_m)^T : \mathbb{R} \rightarrow \mathbb{R}^m \) is called a solution of problem (2.1)–(2.2) if it satisfies Eq. (2.1) for almost all \( t \in (t_0, \infty) \) and conditions (2.2) for \( t \leq t_0 \).
Henceforth assume that conditions (a1)-(a4) hold for problem (2.1)-(2.2) and its modifications, and the problem has a unique solution.

We will use some traditional notations. A matrix $B = (b_{ij})_{j=1}^{m}$ is nonnegative if $b_{ij} \geq 0$ and positive if $b_{ij} > 0$, $i$, $j = 1, \ldots, m$; $\|a\|$ is an arbitrary fixed norm of a column vector $a = (a_1, \ldots, a_m)^T$ in $\mathbb{R}^m$; $\|B\|$ is the corresponding matrix norm of a matrix $B$. $|a| = (|a_1|, \ldots, |a_m|)^T$ and $|B| = (|b_{ij}|)_{j=1}^{m}$.

Problem (2.1)-(2.2) has a unique global solution on $[t_0, \infty)$ if, for example, we assume along with (a1)-(a4) that the functions $F_{ij}(t, u)$ are globally Lipschitz in $u$. The following classical definition of an M-matrix will be used.

**Definition 2.2** [5]. A matrix $B = (b_{ij})_{j=1}^{m}$ is called a (non-singular) M-matrix if $b_{ij} \leq 0$, $i \neq j$ and one of the following equivalent conditions holds:

1. there exists a positive inverse matrix $B^{-1} > 0$;
2. the principal minors of matrix $B$ are positive.

**Definition 2.4.** System (2.1) is globally exponentially stable if there exist constants $M > 0$ and $\lambda > 0$ such that for any solution $X(t)$ of problem (2.1)-(2.2) the inequality

$$
\|X(t)\| \leq Me^{-\lambda(t-t_0)} \left( \|X(t_0)\| + \sup_{t < t_0} \|\varphi(t)\| \right)
$$

holds, where $\varphi(t) = (\varphi_1(t), \ldots, \varphi_m(t))^T$, $M$ and $\lambda$ do not depend on $t_0$.

We define the matrix $C$ as follows

$$
C = (c_{ij})_{j=1}^{m}, \quad c_{ij} = 1 - \frac{A_i(A_i + L_{ij})(m+L_{ij})}{x_i}, \quad c_{ij} = -\frac{A_iL_{ij} + L_{ij}}{x_i}, \quad i \neq j. \quad (2.3)
$$

**Theorem 2.5.** Suppose $C$ defined by (2.3) is an M-matrix. Then system (2.1) is globally exponentially stable.

**Proof.** We extend $\varphi$ to $[t_0, \infty)$ by $\varphi(t) = 0$ and denote the extended function by $\tilde{\varphi}$. The solution $X(t) = (x_1(t), \ldots, x_m(t))^T$ of problem (2.1)-(2.2) is also a solution of the problem

$$
\dot{x}_i(t) = -a_i(t)x_i(h_i(t)) + \sum_{j=1}^{m} F_{ij}(t, x_j(g_{ij}(t))) + \tilde{\varphi}_j(g_{ij}(t)) - a_i(t)\tilde{\varphi}_i(h_i(t)), \quad t \geq t_0 \quad (2.4)
$$

and $x_i(t_0) = 0$, $t < t_0$, $x_i(t_0) = x_i^0$, $i = 1, \ldots, m$. Let $0 < \lambda < \min_i x_i$. The substitution $x_i(t) = e^{-\lambda(t-t_0)}y_i(t)$ for $t \geq t_0$ into system (2.4) yields

$$
\dot{y}_i(t) = \lambda y_i(t) - e^{\lambda(t-t_0)}a_i(t)y_i(h_i(t)) + \sum_{j=1}^{m} e^{\lambda(t-t_0)}F_{ij}(t, e^{-\lambda(t-t_0)}y_j(g_{ij}(t)) + \tilde{\varphi}_j(g_{ij}(t))) - e^{\lambda(t-t_0)}a_i(t)\tilde{\varphi}_i(h_i(t)). \quad (2.5)
$$

Let $\mu_i(t) := e^{\lambda(t-h_i(t))}a_i(t) - \lambda$, then

$$
\dot{y}_i(t) = -\mu_i(t)y_i(t) + e^{\lambda(t-h_i(t))}a_i(t) \int_{h_i(t)}^{t} \dot{y}_i(s)ds + \sum_{j=1}^{m} e^{\lambda(t-t_0)}F_{ij}(t, e^{-\lambda(t-t_0)}y_j(g_{ij}(t)) + \tilde{\varphi}_j(g_{ij}(t)) - e^{\lambda(t-t_0)}a_i(t)\tilde{\varphi}_i(h_i(t)).
$$

For the function $\dot{y}_i(s)$ in the second term, we substitute the right-hand side of Eq. (2.5) and obtain

$$
\dot{y}_i(t) = -\mu_i(t)y_i(t) + e^{\lambda(t-h_i(t))}a_i(t) \times \int_{h_i(t)}^{t} \left[ \dot{y}_i(s) - e^{\lambda(s-h_i(s))}a_i(s)y_i(h_i(s)) + \sum_{j=1}^{m} e^{\lambda(s-t_0)}F_{ij}(s, e^{-\lambda(s-t_0)}y_j(g_{ij}(s)) + \tilde{\varphi}_j(g_{ij}(s)) - e^{\lambda(s-t_0)}a_i(t)\tilde{\varphi}_i(h_i(s)) \right] ds
$$

$$
+ \sum_{j=1}^{m} e^{\lambda(t-t_0)}F_{ij}(t, e^{-\lambda(t-t_0)}y_j(g_{ij}(t)) + \tilde{\varphi}_j(g_{ij}(t))) - e^{\lambda(t-t_0)}a_i(t)\tilde{\varphi}_i(h_i(t)).
$$
Hence
\[ y_i(t) = e^{\int_{t_0}^{t} \mu_j(t) ds} x_i(t_0) \]
\[ + \int_{t_0}^{t} e^{\int_{s}^{t} \mu_j(\zeta) d\zeta} \left( C(0) - e^{\int_{0}^{\zeta} \mu_j(\xi) d\xi} \right) \left( y_i(\zeta) - e^{i\tau_j} a_i(\zeta) y_i(h_i(\zeta)) + \sum_{j=1}^{m} e^{i\tau_k} F_{ij}(\zeta, e^{-i\tau_k} g_j(\zeta)) \right) \right) \]
\[ - e^{i\tau_j} a_i(\zeta) \bar{\phi}_i(h_i(\zeta)) \right) d\zeta \]
\[ + \sum_{j=1}^{m} e^{i\tau_j} F_{ij}(s, e^{-i\tau_j} g_j(s)) y_i(g_i(s)) \right) - e^{i\tau_j} a_i(s) \bar{\phi}_i(h_i(s)) \right) ds. \]

We have
\[ |e^{i\tau_j} F_{ij}(t, e^{-i\tau_j} g_j(t)) y_i(g_i(t)) + \bar{\phi}_i(g_i(t))| \leq e^{\tau_j} L_{ij} e^{-i\tau_j} g_j(t) \right) | + |\bar{\phi}_i(g_i(t))| \]
\[ = L_{ij} e^{i\tau_j} g_j(t) | + e^{i\tau_j} |\bar{\phi}_i(h_i(t))| \]

Since \( t - g_j(t) \leq \sigma \) and \( \bar{\phi}_i(g_i(t)) = 0 \) for \( t \geq \tau + \sigma \), the inequalities
\[ e^{i\tau_j} |\bar{\phi}_i(g_i(t))| \leq e^{i\tau_j} |\bar{\phi}_i(t)|, \]
\[ e^{i\tau_j} |\bar{\phi}_i(h_i(t))| \leq e^{i\tau_j} |\bar{\phi}_i(h_i(t))| \]
hold. Then
\[ |y_i(t)| \leq |x_i(t_0)| + \int_{t_0}^{t} e^{\int_{s}^{t} \mu_j(\zeta) d\zeta} \left( A_{ij} e^{i\tau_j} \int_{h_i(s)}^{h_i(t)} \left( y_i(\zeta) + e^{i\tau_j} a_i(\zeta) y_i(h_i(\zeta)) + \sum_{j=1}^{m} e^{i\tau_j} L_{ij} y_i(g_i(s)) \right) + \sum_{j=1}^{m} e^{i\tau_j} \Phi \right) \mu_j(s) ds, \]
where \( \Phi = \max_{0 \leq s \leq 0} |\bar{\phi}_i| = \max_{0 \leq s \leq 0} |\bar{\phi}_i| \). Let us fix some \( b > t_0 \) and denote \( \bar{y}_i = \max_{0 \leq s \leq 0} |y_i(t)|. \)
\[ \bar{y}_i = \max_{0 \leq s \leq 0} |y_i(t)|. \]

We define the matrix \( C(\lambda) = (c_{ij}(\lambda))_{ij=1}^{m} \) with the entries
\[ c_{ii}(\lambda) = 1 - A_{ii} e^{i\tau_i} + e^{i\tau_j} A_{ij} e^{i\tau_j}, \]
\[ c_{ij}(\lambda) = - A_{ij} e^{i\tau_j} e^{i\tau_i} e^{i\tau_i} L_{ji}, \]
\[ \lambda \neq j. \]

Clearly, the vector inequality \( C(\lambda) \bar{y} \leq |X(t)| + \Phi M(\lambda) \bar{y} \) is valid for \( t_0 \leq t \leq b, \)
\[ M(\lambda) = \left( \frac{\sum_{j=1}^{m} L_{ij} e^{i\tau_i} + A_{ij} e^{i\tau_j}}{\lambda_i - \lambda_j} + 1, \ldots, \frac{\sum_{j=1}^{m} L_{ij} e^{i\tau_i} + A_{ij} e^{i\tau_j}}{\lambda_i - \lambda_j} + 1 \right) \]
and we have \( \lim_{\lambda \to 0} C(\lambda) = C(0) = C \). By the assumption of the theorem, \( C(0) \) is an M-matrix. For \( 0 < \lambda < \min_{0 \leq i \leq m} \lambda_i \), the entries of the matrix \( C(\lambda) \) are continuous functions; therefore, the determinant of this matrix is a continuous function. For some small \( \lambda > 0 \) all the principal minors of \( C(\lambda) \) are positive; the latter along with \( c_{ij}(\lambda) \leq 0, \lambda \neq j \) implies that \( C(\lambda) \) is an M-matrix for small \( \lambda \). If we fix such parameter \( \lambda = \lambda_0 \), then for \( \bar{y} \) there is an a priori estimate
\[ \| \bar{y} \| \leq M\| X(t_0) \| + \Phi, \quad M = \| C^{-1}(\lambda_0) \| \max_{1 \leq t \leq n} \| M(t) (\lambda_0) \|, \]

where \( M \) does not depend on \( b \) and \( t_0 \). Finally, \( X(t) = e^{-i\lambda(t - \lambda_0)} Y(t), \)
\[ \| X(t) \| \leq M\| X(t_0) \| + \Phi e^{-i\lambda(t - \lambda_0)} \]
which completes the proof. \( \square \)

Consider the system with off-diagonal nonlinearities
\[ \dot{x}_i(t) = -a_i(t)x_i(h_i(t)) + \sum_{j \neq i} F_{ij}(t, x_j(g_j(t))), \quad t \geq 0, \quad i = 1, \ldots, m. \] (2.6)
Corollary 1. Suppose that the matrix \( C \) defined by
\[
C = (c_{ij})_{i,j=1}^{m}, \quad c_{ii} = 1 - \frac{A_i^2 \tau_i}{x_i}, \quad c_{ij} = -\frac{A_i A_j \tau_i + L_{ij}}{x_i}, \quad i \neq j
\] (2.7)
is an M-matrix. Then system (2.6) is globally exponentially stable.

The next corollary examines the system with a non-delay linear term
\[
\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{m} F_{ij}(t, x_j(g_j(t))), \quad t \geq 0, \quad i = 1, \ldots, m.
\] (2.8)
Denote
\[
B = (b_{ij})_{i,j=1}^{m}, \quad b_{ii} = 1 - \frac{L_{ii}}{x_i}, \quad b_{ij} = -\frac{L_{ij}}{x_i}, \quad i \neq j.
\] (2.9)

Corollary 2. Suppose \( B \) defined by (2.9) is an M-matrix. Then system (2.8) is globally exponentially stable.

Remark 1. System (2.8) represents the so-called pure-delay case. Interesting and significant results for neural networks models with pure delays were recently obtained in [8,19]. Note that Corollary 2 is a partial case of the results of the paper [19].

For the delay linear system
\[
\dot{x}_i(t) = \sum_{j=1}^{m} a_{ij}(t)x_j(g_j(t)), \quad i = 1, \ldots, m
\] (2.10)
assume that \( a_{ij} \) are essentially bounded on \([0, \infty)\), \( 0 < \alpha_i \leq \tau_i(t) \leq A_i \). \( |a_{ij}(t)| \leq A_{ij}, i \neq j, g_j \) are Lebesgue measurable functions, \( 0 \leq t - g_j(t) \leq \sigma_j \). Denote
\[
D = (d_{ij})_{i,j=1}^{m}, \quad d_{ii} = 1 - \frac{A_i^2 \sigma_i}{x_i}, \quad d_{ij} = -\frac{A_i A_j \sigma_i + A_{ij}}{x_i}, \quad i \neq j.
\] (2.11)

Corollary 3. Suppose \( D \) defined by (2.11) is an M-matrix. Then system (2.10) is exponentially stable.

The same result holds for the linear system with non-delay diagonal terms
\[
\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{m} a_{ij}(t)x_j(g_j(t)), \quad i = 1, \ldots, m,
\] (2.12)
where \( a_{ij} \) are essentially bounded on \([0, \infty)\) functions, \( 0 < \alpha_i \leq a_i(t) \leq A_i \). \( |a_{ij}(t)| \leq A_{ij}, i \neq j, g_j \) are measurable functions, \( 0 \leq t - g_j(t) \leq \sigma_j \). Denote
\[
F = (f_{ij})_{i,j=1}^{m}, \quad f_{ii} = 1, f_{ij} = -\frac{A_{ij}}{x_j}, \quad i \neq j.
\] (2.13)

Corollary 4. Suppose \( F \) defined by (2.13) is an M-matrix. Then system (2.12) is exponentially stable.

Corollary 5. Suppose \( m = 2, A_1(A_1 + L_{11}) \tau_1 + L_{11} < \alpha_1 \) and
\[
(\alpha_1 - A_1(A_1 + L_{11}) \tau_1 - L_{11})(\alpha_2 - A_2(A_2 + L_{22}) \tau_2 - L_{22}) > L_{12}L_{21}(1 + A_1 \tau_1)(1 + A_2 \tau_2).
\] (2.14)
Then system (2.1) is globally exponentially stable.

Proof. For \( m = 2 \) the matrix \( C \) denoted by (2.3) has the form
\[
C = \begin{pmatrix}
1 - \frac{A_1 A_2}{x_1} & -\frac{A_1 A_2 \tau_1 + L_{12}}{x_1} \\
-\frac{A_1 A_2 \tau_1 + L_{12}}{x_2} & 1 - \frac{A_1 A_2}{x_2} \\
\end{pmatrix}.
\]
The off-diagonal entries are negative, by the assumption of the corollary the principal minors are positive, so \( C \) is an M-matrix.

Corollary 6. Suppose \( m = 2, L_{11} < \alpha_1 \) and \((\alpha_1 - L_{11})(\alpha_2 - L_{22}) > L_{12}L_{21}\). Then system (2.8) is globally exponentially stable.
Corollary 7. Suppose $m = 2$, $\sigma_{11} < \alpha_{1}/A_{1}^{2}$ and the inequality
$$
(\alpha_{1} - A_{1}^{2}\sigma_{11})(\alpha_{2} - A_{2}^{2}\sigma_{22}) > A_{12}A_{21}(1 + A_{1}\sigma_{11})(1 + A_{2}\sigma_{22})
$$
holds. Then system (2.10) is exponentially stable.

Corollary 8. Suppose $m = 2$ and $\alpha_{1}\alpha_{2} > A_{12}A_{21}$. Then system (2.12) is exponentially stable.

Remark 2. By Corollary 8, the linear system
$$
\begin{align*}
\dot{x}(t) &= -a_{11}x(t) + a_{12}y(t) \\
\dot{y}(t) &= a_{21}x(t) - a_{22}y(t)
\end{align*}
$$
(2.15)
with the coefficients $a_{ii} > 0$ is exponentially stable if
$$
a_{11}a_{22} > a_{12}a_{21}. \quad (2.16)
$$
Condition (2.16) is necessary and sufficient for exponential stability of system (2.15); therefore, for system (2.15), which is a partial case of (2.1), Theorem 2.5 gives necessary and sufficient exponential stability conditions.

In the paper [10] Gopalsamy considered autonomous system (1.3). For $n = 1$ it has the form
$$
\begin{align*}
\dot{x}(t) &= -a_{1}x(t - \tau_{1}) + a_{12}f_{1}(y(t - \sigma_{1})) \\
\dot{y}(t) &= -a_{2}y(t - \tau_{2}) + a_{12}f_{2}(x(t - \tau_{2}))
\end{align*}
$$
(2.17)
where $a_{1} > 0$, $a_{2} > 0$, $\tau_{i} \geq 0$, $\sigma_{i} \geq 0$, $|f_{i}(u)| \leq L_{i}|u|$, $i = 1, 2$. In [10] the following global attractivity result was obtained: if $a_{i}\tau_{i} < 1$ and
$$
\frac{1 - a_{1}\tau_{1}}{1 + a_{1}\tau_{1}} > \frac{a_{12}L_{1}}{a_{1}}, \quad \frac{1 - a_{2}\tau_{2}}{1 + a_{2}\tau_{2}} > \frac{a_{12}L_{2}}{a_{2}} \quad (2.18)
$$
then any solution of system (2.17) tends to zero. By Corollary 5 Eq. (2.17) is exponentially stable if $a_{i}\tau_{i} < 1$ and
$$
\frac{(1 - a_{1}\tau_{1})(1 - a_{2}\tau_{2})}{(1 + a_{1}\tau_{1})(1 + a_{2}\tau_{2})} > \frac{a_{12}a_{21}L_{1}L_{2}}{a_{1}a_{2}}. \quad (2.19)
$$
Obviously condition (2.18) implies (2.19).

Example 1. Consider system (2.17) where $a_{1} = 0.8$, $a_{2} = 0.5$, $a_{12} = a_{21} = 1$, $\tau_{1} = 0.5$, $\tau_{2} = 0.4$, $|f_{i}(u)| \leq L_{i}|u|$ with $L_{1} = 0.5$, $L_{2} = 0.2$, $\sigma_{i} \geq 0$. Here the first inequality in (2.18) does not hold since
$$
\frac{1 - a_{1}\tau_{1}}{1 + a_{1}\tau_{1}} = \frac{3}{7} < \frac{5}{8} = \frac{a_{12}L_{1}}{a_{1}}
$$
and therefore the result of [10] cannot be applied. However, $a_{1}\tau_{1} = 0.4 < 1$ and inequality (2.19)
$$
\frac{(1 - a_{1}\tau_{1})(1 - a_{2}\tau_{2})}{(1 + a_{1}\tau_{1})(1 + a_{2}\tau_{2})} = \frac{2}{7} > \frac{1}{4} = \frac{a_{12}a_{21}L_{1}L_{2}}{a_{1}a_{2}}
$$
holds, thus Corollary 5 implies exponential stability, hence for $n = 1$ ($m = 2$) we obtained the result which is sharper than the relevant result in [10].

In the next section, we provide more in-depth analysis of systems with leakage delays which include (2.17) as a special case.

3. BAM Network with time-varying delays

In [10] a class (1.3) of BAM neural networks with leakage (forgetting) delays was under study. Via Lyapunov functionals method, sufficient conditions for the existence of a unique equilibrium and its global stability for system (1.3) were obtained. To extend and improve the results of [10,18,20], we will focus on the non-autonomous BAM neural network model
$$
\begin{align*}
\dot{x}_{i}(t) &= r_{i}(t) \left( -a_{i}x_{i}(h_{i}^{(1)}(t)) + \sum_{j=1}^{n} a_{ij}f_{j}(h_{j}^{(2)}(t)) \right) + I_{i} \\
\dot{y}_{i}(t) &= p_{i}(t) \left( -b_{i}y_{i}(h_{i}^{(2)}(t)) + \sum_{j=1}^{n} b_{ij}g_{j}(x_{j}(h_{j}^{(1)}(t))) \right) + J_{i}
\end{align*}
$$
(3.1)
i = 1, \ldots, n, \quad t \geq 0, \quad \text{with the initial conditions}
$$
x_{i}(t) = \varphi_{i}(t), \quad y_{i}(t) = \varphi_{i+1}(t), \quad t < 0, \quad i = 1, \ldots, n. \quad (3.2)
$$
We will say that a norm in $\mathbb{R}^n$ is monotone if $0 \leq a \leq b$ (a componentwise inequality, i.e. $0 \leq a_i \leq b_i$, for all $i$) implies $\|a\| \leq \|b\|$. The following auxiliary lemmas will be used.

**Lemma 3.1** [13, Chapter 2, Section 8.1], [9, Chapter 4, Problem 228]. For any linear operator $L : \mathbb{R}^m \to \mathbb{R}^m$ and $\varepsilon > 0$ there exists a norm in $\mathbb{R}^m$ such that for the spectral radius $r(L)$ of $L$ we have

$$r(L) > \|L\| - \varepsilon. \tag{3.3}$$

If the operator $L$ is positive then the norm in $\mathbb{R}^m$ is monotone.

Consider the algebraic system

$$u_i = \sum_{j=1}^{m} F_{ij}(u_j), \quad i = 1, \ldots, m. \tag{3.4}$$

where $|F_{ij}(u) - F_{ij}(v)| \leq L_{ij}|u - v|$.

**Lemma 3.2.** Let $r(L)$ be a spectral radius of the matrix $L = (L_{ij})_{i,j=1}^{m}$. If $r(L) < 1$ then system (3.4) has a unique solution.

**Proof.** Consider the operator $T : \mathbb{R}^m \to \mathbb{R}^m$ denoted by

$$T(u) := T((u_1, \ldots, u_m)^T) = \left( \sum_{j=1}^{m} F_{1j}(u_j), \ldots, \sum_{j=1}^{m} F_{mj}(u_j) \right)^T. $$

Then

$$|T(u) - T(v)| \leq \left( \sum_{j=1}^{m} L_{ij}|u_j - v_j|, \ldots, \sum_{j=1}^{m} L_{mj}|u_j - v_j| \right)^T = L|u - v|,$$

where the inequality is understood componentwise.

Since $r(L) < 1$ and the operator $L$ is positive, by Lemma 3.1 we can choose a monotone norm in $\mathbb{R}^m$ such that the corresponding norm $\|L\| \leq q < 1$. We fix now such a norm and have

$$\|T(u) - T(v)\| \leq L\|u - v\| \leq \|L\|\|u - v\| \leq q\|u - v\|.$$

By the Banach contraction principle the equation $u = T(u)$ has a unique solution. \hfill $\square$

**Corollary 9.** Suppose at least one of the following conditions holds:

1. $\max 2|\lambda(L)| < 1$, where the maximum is taken over all eigenvalues of matrix $L$;
2. $\max_{i} \sum_{j} L_{ij} < 1$;
3. $\max_{i} \sum_{j} L_{ij} < 1$;
4. $\sum_{i,j} \sum_{j} L_{ij} < 1$.

Then system (3.4) has a unique solution.

It should be noted that the proof of Lemma 3.2 is original and shorter than, for example, the recently published proof [25, Theorem 2.2].

Henceforth, assume that the following assumptions hold for (3.1)–(3.2):

- (b1) $r_i$, $p_i$ are Lebesgue measurable essentially bounded on $[0, \infty)$ functions, $0 < r_i \leq r_i(t) \leq R_i, \ 0 < p_i \leq p_i(t) \leq P_i$;
- (b2) $f_j(\cdot)$, $g_j(\cdot)$ are continuous functions; $|f_j(u) - f_j(v)| \leq L_j^f|u - v|$, $|g_j(u) - g_j(v)| \leq L_j^g|u - v|$;
- (b3) $h_j^{(1)}, \ h_j^{(2)}, \ h_j^{(1)(1)}, \ h_j^{(2)(1)}$ are Lebesgue measurable functions, $0 \leq t - h_j^{(1)}(t) \leq \tau_j^{(1)}$, $0 \leq t - h_j^{(2)}(t) \leq \tau_j^{(2)}$, $0 \leq t - h_j^{(1)(1)}(t) \leq \sigma_j^{(1)}$, $0 \leq t - h_j^{(2)(1)}(t) \leq \sigma_j^{(2)}$;
- (b4) $\varphi_i$ are continuous functions.

Let $(\xi_1, \ldots, \xi_n, \ y_1, \ldots, y_n)$ be a solution of the system

$$a_i\xi_i = \sum_{j=1}^{n} a_{ij}f_j(y_j) + I_i \tag{3.5}$$

Let us transform system (3.5) into the following one

$$b_jy_j = \sum_{i=1}^{n} b_{ij}g_i(\xi_i) + J_i.$$
\[ x_i = \sum_{j=1}^{n} \frac{a_{ij}}{a_i} f_j(y_j) + \frac{I_i}{a_i} \]
\[ y_i = \sum_{j=1}^{n} \frac{b_{ij}}{b_i} g_j(x_j) + \frac{J_i}{b_i}. \]

(3.6)

Denoting \( u_j = x_j, \ j = 1, \ldots, n, \ u_i = y_{j-n}, \ j = n+1, \ldots, 2n, \)
\[ F_j(u) = \begin{cases} a_{ij-n} f_j(u) + \frac{I_j}{a_{ij-n}}, \ i = 1, \ldots, n; \ j = n+1, \ldots, 2n, \\ b_{ij-n} g_j(u) + \frac{J_j}{b_{ij-n}}, \ i = n+1, \ldots, 2n; \ j = 1, \ldots, n, \\ 0, \ otherwise. \end{cases} \]

we can rewrite system (3.6) in the form of (3.4) with \( m = 2n, \ |F_j(u) - F_j(v)| \leq L_i |u - v|. \)

We introduce the matrix \( A = (L_{ij})_{i=1}^{2n} \):
\[
A = \begin{pmatrix}
0 & \cdots & 0 & \frac{|a_{11}| I_1}{a_1} & \cdots & \frac{|a_{n1}| I_1}{a_1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{|a_{1n}| I_1}{a_n} & \cdots & \frac{|a_{nn}| I_1}{a_n}
\end{pmatrix}
\]

By the same token we can rewrite (3.5) in the form (3.4), where
\[ F_j(u) = \begin{cases} a_{ij-n} f_j(u) + \frac{I_j}{a_{ij-n}}, \ i = 1, \ldots, n; \ j = n+1, \ldots, 2n, \\ b_{ij-n} g_j(u) + \frac{J_j}{b_{ij-n}}, \ i = n+1, \ldots, 2n; \ j = 1, \ldots, n, \\ 0, \ otherwise. \end{cases} \]

with \( m = 2n, \ |F_j(u) - F_j(v)| \leq L_i |u - v|, \) and introduce the matrix \( B = (L_i)_{j=1}^{2n} \):
\[
B = \begin{pmatrix}
0 & \cdots & 0 & \frac{|b_{11}| I_1}{b_1} & \cdots & \frac{|b_{n1}| I_1}{b_1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{|b_{1n}| I_1}{b_n} & \cdots & \frac{|b_{nn}| I_1}{b_n}
\end{pmatrix}
\]

In the following theorem we apply Lemma 3.2 to systems (3.5) and (3.6) with \( L = A \) and \( L = B, \) and obtain conditions 1–4 and 5–8, respectively.

**Theorem 3.3.** Suppose at least one of the following conditions holds:

1. \( \max |\lambda(A)| < 1, \) where maximum is taken on all eigenvalues of matrix \( A.\)
2. \( \max \sum_{j=1}^{n} \frac{|a_{ij}| I_j}{a_i} < 1, \) \( \max \sum_{j=1}^{n} \frac{|b_{ij}| I_j}{b_i} < 1.\)
3. \( \max \sum_{j=1}^{n} \frac{|a_{ij}| I_j}{a_i} < 1, \) \( \max \sum_{j=1}^{n} \frac{|b_{ij}| I_j}{b_i} < 1.\)
4. \( \sum_{j=1}^{n} \sum_{j=1}^{n} \left( \frac{|a_{ij}| I_j}{a_i} \right)^2 + \left( \frac{|b_{ij}| I_j}{b_i} \right)^2 < 1.\)
5. \( \max |\lambda(B)| < 1, \) where maximum is taken on all eigenvalues of matrix \( B.\)
6. \( \max \sum_{j=1}^{n} \frac{|a_{ij}| I_j}{b_i} < 1, \) \( \max \sum_{j=1}^{n} \frac{|b_{ij}| I_j}{b_i} < 1.\)
7. \( \max \sum_{j=1}^{n} \frac{|a_{ij}| I_j}{b_i} < 1, \) \( \max \sum_{j=1}^{n} \frac{|b_{ij}| I_j}{b_i} < 1.\)
8. \( \sum_{j=1}^{n} \sum_{j=1}^{n} \left( \frac{|a_{ij}| I_j}{b_i} \right)^2 + \left( \frac{|b_{ij}| I_j}{b_i} \right)^2 < 1.\)

Then system (3.5) has a unique solution and thus system (3.1) has a unique equilibrium.
Remark 3. Note that the conclusion of Theorem 3.3 under condition 7 was obtained in paper [10].

Below we assume everywhere that system (3.1) has a unique equilibrium \((x^*, y^*)\). To obtain a global stability condition for this equilibrium, consider the matrix \(C_{\text{bam}} = (c_{ij})_{n \times n} \), where

\[
c_u = \begin{cases} 
1 - a_i R_i^2 \frac{t_i}{r_i}, & i = 1, \ldots, n, \\
1 - b_{i-n} P_{i-n}^2 \frac{t_{i-n}}{\tilde{p}_{i-n}}, & i = n + 1, \ldots, 2n,
\end{cases} \tag{3.7}
\]

\[
c_y = \begin{cases} 
-|a_{i-n}| R_i^L \frac{R_i^L}{r_i} (a_i R_i^2 t_i^{(1)} + 1) / (r_i a_i), & i = 1, \ldots, n, \ j = n + 1, \ldots, 2n, \\
-|b_{i-n}| P_i^L \frac{P_i^L}{\tilde{p}_i} (b_i P_i^2 t_i^{(2)} + 1) / (\tilde{p}_{i-n} b_{i-n}), & i = n + 1, \ldots, 2n, \ j = 1, \ldots, n, \\
0, & \text{otherwise}.
\end{cases} \tag{3.8}
\]

Theorem 3.4. Suppose matrix \(C_{\text{bam}}\) is an M-matrix. Then the equilibrium \((x^*, y^*)\) of system (3.1) is globally exponentially stable.

Proof. After the substitution \(x_i(t) = u_i(t) + x_i^*, \ y_i(t) = v_i(t) + y_i^*, \) system (3.1) has the form

\[
\begin{align*}
\dot{u}_i(t) &= -r_i(t) a_i u_i(h_i^{(1)}(t)) + \sum_{j=1}^{n} a_{ij} r_i(t) \left( f_j(v_j(t)) + y_j^* - f_j(y_j^*) \right) \\
\dot{v}_i(t) &= -p_i(t) b_i v_i(h_i^{(2)}(t)) + \sum_{j=1}^{n} b_{ij} p_i(t) \left( g_j(u_j(t)) + x_j^* - g_j(x_j^*) \right),
\end{align*} \tag{3.9}
\]

where

\[
x_i(t) = \begin{cases} 
u_i(t), & i = 1, \ldots, n, \\
v_{i-n}(t), & i = n + 1, \ldots, 2n,
\end{cases} \quad
a_i(t) = \begin{cases} 
r_i(t)a_i, & i = 1, \ldots, n, \\
\tilde{p}_{i-n}(t)b_{i-n}, & i = n + 1, \ldots, 2n,
\end{cases} \quad
h_i(t) = \begin{cases} 
h_i^{(1)}(t), & i = 1, \ldots, n, \\
h_i^{(2)}(t), & i = n + 1, \ldots, 2n,
\end{cases} \quad
g_{ij}(t) = \begin{cases} 
0, & \text{otherwise},
\end{cases}
\]

\[
F_{ij}(t,x) = \begin{cases} 
a_{i-j-n} r_i(t) \left( f_{j-n}(x + y_{j-n}^*) - f_{j-n}(y_{j-n}^*) \right), & i = 1, \ldots, n, \ j = n + 1, \ldots, 2n, \\
b_{i-j-n} P_{i-n}(t) \left( g_j(x + x_j^*) - g_j(x_j^*) \right), & i = n + 1, \ldots, 2n, \ j = 1, \ldots, n, \\
0, & \text{otherwise}.
\end{cases}
\]

We have \(0 < \gamma_i \leq a_i(t) \leq A_i\), where

\[
\gamma_i = \begin{cases} 
r_i a_i, & i = 1, \ldots, n, \\
\tilde{p}_{i-n} b_{i-n}, & i = n + 1, \ldots, 2n,
\end{cases} \quad
A_i = \begin{cases} 
R a_i, & i = 1, \ldots, n, \\
P_{i-n} b_{i-n}, & i = n + 1, \ldots, 2n
\end{cases}
\]

and \(|F_{ij}(t,x)| \leq L_{ij} |x|\) with the constant

\[
L_{ij} = \begin{cases} 
|a_{i-j-n}| R_i^L, & i = 1, \ldots, n, \ j = n + 1, \ldots, 2n, \\
|b_{i-j-n}| P_i^L, & i = n + 1, \ldots, 2n, \ j = 1, \ldots, n, \\
0, & \text{otherwise}.
\end{cases}
\]

System (3.9) with \(m = 2n\) has form (2.6), where matrix \(C_{\text{bam}}\) corresponds to matrix \(C\) defined by (2.7). All conditions of Corollary 1 hold; therefore, the trivial solution of system (3.9) is globally exponentially stable; hence the equilibrium \((x^*, y^*)\) of system (3.1) is globally exponentially stable. \(\square\)

Corollary 10. Suppose at least one of the following conditions holds:

1. \(\sum_{j=1}^{n} \frac{|a_{ij}| R_i^L (a_i R_i^2 t_i^{(1)} + 1)}{r_i a_i} < 1 - \frac{a_i R_i^2 t_i^{(1)}}{r_i},\)
2. \(\sum_{j=1}^{n} \frac{|b_{ij}| P_i^L (b_i P_i^2 t_i^{(2)} + 1)}{p_i b_i} < 1 - \frac{b_i P_i^2 t_i^{(2)}}{p_i}, i = 1, \ldots, n.\)
2. \[ \sum_{i=1}^{n} \frac{|a_i|R_i^{L_i}}{r_i a_i} (a_i R_i \tau_i^{(1)} + 1) < 1 - \frac{b_j P_j^2 \tau_j^{(2)}}{P_j}, \]
\[ \sum_{i=1}^{n} \frac{|b_j| P_j L_i^{L_i}}{p_i b_i} (b_j P_j \tau_j^{(1)} + 1) < 1 - \frac{a_i R_i^2 \tau_i^{(1)}}{r_i}, \quad j = 1, \ldots, n. \]

3. There exist positive numbers \( \mu_k, \ k = 1, \ldots, 2n \) such that
\[ \sum_{i=1}^{n} \frac{\mu_{i+k} |a_i|R_i^{L_i}}{r_i a_i} (a_i R_i \tau_i^{(1)} + 1) < \mu_i \left( 1 - \frac{a_i R_i^2 \tau_i^{(1)}}{r_i} \right), \]
\[ \sum_{i=1}^{n} \frac{\mu_{i+k} |b_j| P_j L_i^{L_i}}{p_i b_i} (b_j P_j \tau_j^{(1)} + 1) < \mu_{i+k} \left( 1 - \frac{b_j P_j^2 \tau_j^{(2)}}{P_j} \right), \quad i = 1, \ldots, n. \]

4. There exist positive numbers \( \mu_k, \ k = 1, \ldots, 2n \) such that
\[ \sum_{i=1}^{n} \frac{\mu_{i+k} |a_i|R_i^{L_i}}{r_i a_i} (a_i R_i \tau_i^{(1)} + 1) < \mu_i \left( 1 - \frac{a_i R_i^2 \tau_i^{(1)}}{r_i} \right), \]
\[ \sum_{i=1}^{n} \frac{\mu_{i+k} |b_j| P_j L_i^{L_i}}{p_i b_i} (b_j P_j \tau_j^{(1)} + 1) < \mu_{i+k} \left( 1 - \frac{b_j P_j^2 \tau_j^{(2)}}{P_j} \right), \quad j = 1, \ldots, n. \]

Then the equilibrium \((x^*, y^*)\) of system (3.1) is globally exponentially stable.

**Proof.** By Lemma 2.3 any of the conditions 1–4 implies that \( C_{RBM} \) is an \( M \)-matrix. \( \Box \)

**Remark 4.** Part 3 of Corollary 10 coincides with [18, Theorem 3.1] in the case when \( r_i(t) \) and \( p_j(t) \) are constants. In addition to being more general than [18, Theorem 3.1], the result of Theorem 3.4 does not require to find some positive constants, i.e., the check of the signs of principal minors will immediately indicate whether such constants exist or not.

In the following statement consider system (3.1) without delays in the leakage terms.

**Corollary 11.** Suppose \( h^{(1)}_i(t) \equiv t, \ h^{(2)}_i(t) \equiv t, \) and at least one of the following conditions holds:

1. \[ \sum_{i=1}^{n} \frac{|a_i|R_i^{L_i}}{r_i a_i} < 1, \sum_{i=1}^{n} \frac{|b_j| P_i L_i^{L_i}}{p_i b_i} < 1, \quad i = 1, \ldots, n. \]

2. \[ \sum_{i=1}^{n} \frac{|a_i|R_i^{L_i}}{r_i a_i} < 1, \sum_{i=1}^{n} \frac{|b_j| P_i L_i^{L_i}}{p_i b_i} < 1, \quad j = 1, \ldots, n. \]

3. There exist positive numbers \( \mu_k, \ k = 1, \ldots, 2n \) such that
\[ \sum_{i=1}^{n} \frac{\mu_{i+k} |a_i|R_i^{L_i}}{r_i a_i} < \mu_i \sum_{i=1}^{n} \frac{\mu_{i+k} |a_i|R_i^{L_i}}{r_i a_i} < \mu_{i+k}, \quad i = 1, \ldots, n. \]

4. There exist positive numbers \( \mu_k, \ k = 1, \ldots, 2n \) such that
\[ \sum_{i=1}^{n} \frac{\mu_{i+k} |a_i|R_i^{L_i}}{r_i a_i} < \mu_i \sum_{i=1}^{n} \frac{\mu_{i+k} |a_i|R_i^{L_i}}{r_i a_i} < \mu_{i+k}, \quad j = 1, \ldots, n. \]

Then the equilibrium \((x^*, y^*)\) of system (3.1) is globally exponentially stable.

Consider system (3.1) with \( n = 1 \):
\[ \begin{align*}
\dot{x}(t) &= r(t)(-ax(h_1(t)) + Af(y(l_2(t))) + 1), \\
\dot{y}(t) &= p(t)(-ay(h_2(t)) + Bf(x(l_1(t))) + f).
\end{align*} \tag{3.10} \]

where
\[ a > 0, \quad b > 0, \quad 0 < r < r(t) \leq R, \quad 0 < p \leq p(t) \leq P, \quad |f(u) - f(v)| \leq L |u - v|, \]
\[ |g(u) - g(v)| \leq L |u - v|, \quad 0 \leq t - h_i(t) \leq \tau_i, \quad 0 \leq t - l_i(t) \leq \sigma_i, \quad i = 1, 2. \]

**Corollary 12.** Suppose \( \frac{aR^2 \tau_1}{r} < 1, \) and
\[ \frac{ABPPL^2(aR \tau_1 + 1)(aP \tau_2 + 1)}{r P a b} < \left( 1 - \frac{aR^2 \tau_1}{r} \right) \left( 1 - \frac{bP^2 \tau_2}{P} \right). \]

Then the equilibrium \((x^*, y^*)\) of system (3.10) is globally exponentially stable.
Example 2. Consider the particular case of BAM network described by (3.10)
\[
\begin{align*}
\dot{x}(t) &= (20 + \mu \sin t) \left[ -x \left( t - \frac{1 + |\sin t|}{200} \right) + \frac{1}{720} \gamma (t - 3 \sin^2(t)) + 10000 \right] \\
\dot{y}(t) &= (40 + \mu \cos t) \left[ -y \left( t - \frac{1 + |\cos t|}{200} \right) + \frac{1}{200} \gamma (t - 2 \sin^2(t)) + 20000 \right]
\end{align*}
\]  
(3.11)
for \( \mu \geq 0 \). Here \( l^1 = l^2 = a = b = 1 \), \( \bar{r} = 20 - \mu \), \( R = 20 + \mu \), \( p = 4 - \mu \), \( P = 40 + \mu \), \( \tau_1 = \tau_2 = \frac{1}{1000} \), \( A = \frac{720}{20} \), \( B = \frac{1}{100} \).

By Corollary 12, system (3.11) is exponentially stable if \( 1 - \frac{(20 + \mu)^2}{1000(20 - \mu)} - \frac{(40 + \mu)^2}{1000(40 - \mu)} \) < 1 and
\[
\frac{(20 + \mu + 1)(40 + \mu + 1)}{720 \cdot 200(20 - \mu)(40 - \mu)} < \left( 1 - \frac{(20 + \mu)^2}{1000(20 - \mu)} \right) \left( 1 - \frac{(40 + \mu)^2}{1000(40 - \mu)} \right),
\]
which is satisfied, for example, if \( 0 \leq \mu < 18 \).

Note that for \( \mu = 0 \) this example coincides with [18, Example 4.1]. It was also mentioned in [18] that exponential stability of (3.11) cannot be obtained using the results of [10,15,16,23], since leakage delays in (3.11) are time-variable. Therefore, our results for the case \( \mu > 0 \) and with time-varying coefficients and delays are new and applicable to more general models.

4. Discussion and open problems

To obtain sufficient stability conditions for nonlinear delay systems, four different approaches might be used: construction of Lyapunov functionals, application of fixed point theory, either development of estimations for matrix or operator norms, or making use of some special matrix (M-matrix) properties, and the transformations of a given nonlinear system to an operator equation with a Volterra casual operator. While the Lyapunov direct method has been and remains the leading technique, numerous difficulties with the theory and applications to specific systems persist. One of the problems with using the fixed point techniques is the construction of an appropriate map (integral equation) that is sometimes quite difficult or impossible. The technique used in papers [15–17,24] is based on the construction of Lyapunov functionals.

Perhaps it is worth mentioning that the method applied here to nonlinear system (2.1) is somehow related to the approach used in [11] for linear system (2.10); however, to the best of our knowledge, there are no similar results for (2.1).

Remark 2, Example 1, Theorem 3.4 and its corollaries improve and extend results previously obtained for BAM neural networks in [6,10,18,23,28,29].

In the framework of this paper, we could not consider all the applications of the M-matrix method to specific models; therefore we outline some particular cases and extensions that might be of interest for scientists who plan to start future research in this field.

1. Find global stability conditions of system (2.1) for the special cases:
\[
F_{\tilde{y}}(t,x) = \tanh (x_i(t)) \quad \text{and} \quad F_{\tilde{y}}(t,x) \equiv 1 - \frac{2}{1 + e^{-x_i(t)}}.
\]

2. Study global stability for a more general than (2.1) model:
\[
\dot{x}_i(t) = -a_i(t)x_i(h_i(t)) + \sum_{j=1}^{m} \sum_{k=1}^{s} F_{g_k}(t,x_j(g_k(t))), \quad t \geq 0, \quad i = 1, \ldots, m.
\]

3. Derive sufficient stability tests for the equation with a distributed transmission delay
\[
\dot{x}_i(t) = -a_i(t)x_i(h_i(t)) + \sum_{j=1}^{m} \int_{t-\tau_i}^{t} K_{g_k}(t,s)F_{g_k}(s,x_j(g_k(s))) ds, \quad t \geq 0, \quad i = 1, \ldots, m.
\]

4. Investigate stability of the system with distributed delays in all the terms
\[
\dot{x}_i(t) = -a_i(t) \int_{t-\tau_i}^{t} x_i(h_i(s)) d_R_i(t,s) + \sum_{j=1}^{m} \int_{t-\tau_i}^{t} F_{g_k}(s,x_j(g_k(s))) d_T_{g_k}(t,s), \quad t \geq 0, \quad i = 1, \ldots, m.
\]

5. Obtain sufficient stability conditions for the system with an infinite distributed delay
\[
\dot{x}_i(t) = -a_i(t)x_i(h_i(t)) + \sum_{j=1}^{m} \int_{t-\infty}^{t} K_{g_k}(t,s)F_{g_k}(s,x_j(g_k(s))) ds, \quad t \geq 0, \quad i = 1, \ldots, m,
\]

where \( |K_{g_k}(t,s)| \leq Me^{-(t-s)} \). Generalize this result to the case of exponentially decaying infinite leakage delays as well.

6. Analyze global asymptotic stability conditions of (2.1) when condition (a3) is not satisfied but \( \lim_{t \to \infty} h_i(t) = \infty, \lim_{t \to \infty} g_k(t) = \infty \), e.g., the pantograph-type delays \( h_i(t) = \lambda_i t \) for \( 0 < \lambda_i < 1 \). Is it possible to estimate the rate of convergence for some classes of delays?
7. Under which conditions will solutions of BAM system (3.1) with \( l > 0, \ j > 0 \) and positive initial functions be permanent (positive, bounded and separated from zero)?

8. Apply the \( M \)-matrix method to the following generalization of (2.1)

\[
x_i(t) = -a_{ii}(t)x_i(h_i(t)) + \sum_{j=1}^{m} F_{ij}(t, x_1(g_{ij}^{(1)}(t)), \ldots, x_m(g_{ij}^{(m)}(t))), \quad t \geq 0, \ i = 1, \ldots, m.
\]

9. Conjecture:

If \( C \) defined by (2.3) has negative off-diagonal entries \( a_{ij} \leq 0, \ i \neq j \), and its Moore-Penrose pseudoinverse matrix is non-negative then system (2.1) is stable.

**Remark 5.** To apply the results of the present paper, first use [4, Theorem 9] to reduce exponential stability of equations with distributed delays to exponential stability of equations with concentrated delays. For some other methods see recent papers [1–3] and references therein.

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**References**


